Chapter

On Some Important Ordinary Differential Equations of Dynamic Economics

Anastasios Tsoularis

Abstract

Mathematical modeling in economics became central to economic theory during the decade of the Second World War. The leading figure in that period was Paul Anthony Samuelson whose 1947 book, *Foundations of Economic Analysis*, formalized the problem of dynamic analysis in economics. In this brief chapter some seminal applications of differential equations in economic growth, capital and business trade cycles are outlined in deterministic setting. Chaos and bifurcations in economic dynamics are not considered. Explicit analytical solutions are presented only in relatively straightforward cases and in more complicated cases a path to the solution is outlined. Differential equations in modern dynamic economic modeling are extensions and modifications of these classical works. Finally we would like to stress that the differential equations presented in this chapter are of the "stand-alone" type in that they were solely introduced to model economic growth and trade cycles. Partial differential equations such as those which arise in related fields, like Bioeconomics and Differential Games, from optimizing the Hamiltonian of the problem, and stochastic differential equations of Finance and Macroeconomics are not considered here.

Keywords: Walrassian condition, Marshallian condition, homogeneous function, Cobb–Douglas form, endogenous growth

1. Introduction

Ordinary differential equations are ubiquitous in the physical sciences and are fundamental for the understanding of complex engineering systems [1]. In economics they are used to model for instance, economic growth, gross domestic product, consumption, income and investment whereas in finance stochastic differential equations are indispensable in modeling asset price dynamics and option pricing. The vast majority of the ordinary differential equations in economic are autonomous differential equations or difference equations, where time is an implicit variable, whereas the more difficult to solve delay (differential-difference) equations have received much less attention. Difference equations seem a more natural choice of modeling economic processes as key economic variables are monitored at discrete time units but they can present significant complications in their asymptotic behavior and are thus more difficult to analyse. Differential equations on the other hand, can be more amenable to asymptotic stability analysis. Partial differential equations, usually of the second order, for functions of at least two variables arise naturally in modern macroeconomics from solving an optimization problem formulated in a stochastic setting and in optimal control theory. Two books that are recommended for delving deeper into the- economic applications of differential equations are the introductory one by Gandolfo [2] and the more advanced by Brock and Malliaris [3]. Both books are excellent sources for ordinary differential equations in economic dynamics. A more recent book which requires strong mathematical background is by Acemoglu [4].

2. Some differential equations of neoclassical growth theory and business cycles

Some of the most important differential equations developed by economists during a period spanning over sixty years are presented in this section. Most of them beginning with Solow's development of a growth model, which was partly motivated by the works of Harrod and Domar, are models from Neoclassical Growth Theory. The main postulate of Neoclassical Growth Theory is that *economic growth* is driven by three elements: labour, capital, and technology. Economic growth is an important topic in economics and Solow's growth model is the first topic taught in undergraduate economics because of its underlying simplicity and importance as argued by Acemoglu [5]. The differential equation by Samuelson is concerned with demand and supply scenarios. Phillips' work is the earliest attempt to employ classical feedback control theory in order to steer a national economy towards a desired target. The remaining works are differential equations with time lags inherently present in production and capital accumulation. Due to space limitations, the exposition is somewhat uneven with full mathematical analyses of most models and cursory treatments of those with time lags. The choice of the differential equations presented in this chapter is a judicious one, the list is by no means exhaustive, but is meant to afford a glimpse into how the mathematical thinking of some famous economists has influenced the economic growth theory in the twentieth century.

2.1 Harrod-Domar

The Harrod-Domar model was developed independently by Roy Harrod [6] and Evsey Domar [7] to analyze business cycles originally but later was used to explain an economy's growth rate through savings and capital productivity. Output, Y, is a function of capital stock, K, $Y = F(K)$, and the marginal productivity, $\frac{dY}{dK} = c = constant$. The model postulates that the output growth rate is given by

$$\frac{1}{Y}\frac{dY}{dt} = sc - \delta,$$

where s is the savings rate, and δ the capital depreciation rate. The straightforward solution,

$$Y(t) = Y_0 e^{(sc-\delta)t}.$$

clearly demonstrates that increasing investment through savings and productivity boosts economic growth but does not take into account labour input and population size.

2.2 Samuelson

In his 1941 Paul Samuelson [8] paper employed simple differential equations to investigate the stability of equilibrium for several demand–supply scenarios.

The simplest stability analysis was carried out under the Walrasian and Marshallian assumptions. In the former price increases (decreases) if excess demand is positive (negative), whereas in the latter quantity increases (decreases) if excess demand price is positive (negative). Excess demand is the difference between the quantity that buyers are willing to buy and the quantity that suppliers are willing to supply at the same price. Excess demand price is the difference between the price that buyers are willing to pay for a given quantity and the price required by the suppliers.

Let $D(p,\alpha)$ and $S(p)$ denote the demand and supply functions of price, p, respectively with α a shift parameter representing "taste". At equilibrium, price, p^*, and quantity, q^*, are given by

$$q^* = D(p^*,\alpha) = S(p^*)$$

$$\frac{\partial D}{\partial \alpha} > 0, \frac{\partial D}{\partial p} < 0.$$

It is the task of comparative statics to show the determination of the equilibrium values of price and quantity and their sensitivity on the "taste" parameter, α.

The dynamic formulation of the Walrasian assumption is

$$\frac{dp}{dt} = f(D(p) - S(p)), f(0) = 0, f'(0) > 0.$$

Retaining the first order term in a Taylor series expansion near the equilibrium, p^*, we obtain the following linear differential equation

$$\frac{dp}{dt} = a_0 \left(\frac{dD}{dp} - \frac{dS}{dp}\right)_{p^*} (p - p^*),$$

with solution for an initial price, p_0

$$p(t) = p^* + (p^* - p_0) e^{a_0 t \left(\frac{dD}{dp} - \frac{dS}{dp}\right)_{p^*}}.$$

The equilibrium is stable if $\left(\frac{dD}{dp}\right)_{p^*} < \left(\frac{dS}{dp}\right)_{p^*}$. Price must rise when demand increases.

The dynamic formulation of the Marshallian assumption is

$$\frac{dq}{dt} = g(p_D(q) - p_S(q)), g(0) = 0, g'(0) > 0.$$

Neglecting high order terms and using the trivial elementary calculus result, $\frac{dp_D}{dq} = \frac{1}{\frac{dD}{dp}}, \frac{dp_S}{dq} = \frac{1}{\frac{dS}{dp}}$, we obtain

$$q(t) = q^* + (q^* - q_0) \exp\left[b_0 t \left(\frac{1}{\frac{dD}{dp}} - \frac{1}{\frac{dS}{dp}}\right)_{q^*} \right].$$

The equilibrium is stable if $\left(\frac{1}{\frac{dD}{dp}}\right)_{q^*} < \left(\frac{1}{\frac{dS}{dp}}\right)_{q^*}$. Quantity supplied must rise when demand increases, while the change in price is dependent upon the algebraic sign of the supply curve's slope.

2.3 Solow

Robert Solow [9] proposed a growth equation incorporating production, capital growth and growth in the labour force absent from the Harrod-Domar model.

i. Production function: $= F(K, L)$, the quantity of goods by K units of capital and L units of labour at time t. In a closed economy where all output is invested or consumed,

$$Y(t) = C(t) + I(t),$$

where $C(t)$ and $I(t)$ are the consumption and investment functions respectively. An important assumption of the model are the Inada conditions [10]

$$\frac{\partial F}{\partial K} > 0, \frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0.$$

In the limits.

$$\lim_{K \to 0} \frac{\partial F}{\partial K} = \infty, \lim_{L \to 0} \frac{\partial F}{\partial L} = \infty, \lim_{K \to \infty} \frac{\partial F}{\partial K} = 0, \lim_{L \to \infty} \frac{\partial F}{\partial L} = 0.$$

The Inada conditions ensure that F is strictly concave with slope decreasing from infinity to zero.

The function F is linearly homogeneous of degree 1 in K and L (in economic terms this is known as constant returns to scale, increasing capital and labour by a certain amount, results in a proportional rise of production) if

$$Y = F(\alpha K, aL) = \alpha F(K, L), \forall \alpha > 0.$$

In particular, choosing $\alpha = \frac{1}{L}$ and set $y = \frac{Y}{L}, k = \frac{K}{L}$, representing the output and capital per worker respectively

$$\frac{Y}{L} = y = F\left(\frac{K}{L}, 1\right) = f(k).$$

The production function is expressed in terms of a unit of labour and the capital to labour ratio. The assumption of constant returns to scale allows the simplified function, $f(k)$.

ii. Growth of Capital in Economy: The growth of the capital stock, K, is equivalent to growth in investment, I, which is used to increase capital subject to depreciation. Depreciation of capital stock will be accounted for so that I is essentially

$$\text{investment} = \text{rate of change of capital} + \text{capital depreciation rate}$$

or

$$I(t) = \frac{dK}{dt} + \delta K(t),$$

where δ is the constant capital depreciation rate.

Letting $c(t)$ and $i(t)$ denote the consumption and investment per labour unit

$$c(t) = \frac{C}{L}, i(t) = \frac{I}{L},$$
$$y(t) = c(t) + i(t) = c(t) + \frac{1}{L}\frac{dK}{dt} + \delta k = c(t) + \frac{dk}{dt} + \left(\delta + \frac{1}{L}\frac{dL}{dt}\right)k.$$

iii. Growth of the Labour Force with full employment: The assumption in the labour market is that the labour supply is equivalent to the population. There is no unemployment and the growth of labour as function of time follows an exponential growth pattern:

$$L = L_0 e^{nt}.$$

The fundamental differential equation of economic growth is then

$$\frac{dk}{dt} = f(k) - (\delta + n)k - c(t).$$

The differential equations and production functions outlined in these three assumptions are the fundamental elements for Solow's basic differential equation. In Solow's paper, a constant fraction of income is allocated to savings, in particular, $= y(t) - c(t) = f(k) - (1 - s)f(k) = sf(k)$, so that

$$\frac{dk}{dt} = sf(k) - (\delta + n)k.$$

The equilibrium solution to the basic differential equation is found from $sf(k) = (\delta + n)k$. A well-known function is the Cobb–Douglas production function, $Y(K, L) = \alpha K^{\beta} L^{1-\beta}$, $0 < \beta < 1$, where β is the elasticity of output, $\frac{K}{Y}\frac{\partial Y}{\partial K}$, with respect to capital. The use of the Cobb–Douglas production function is justified because it exhibits constant returns to scale: If capital and labour are both increased by the same factor, $\lambda > 1$, output will be increased by exactly the same proportion, $Y(K, L) = \lambda\left(\alpha K^{\beta} L^{1-\beta}\right)$. Also the marginal product, $\frac{\partial Y}{\partial K}$, $\frac{\partial Y}{\partial L}$, diminishes as either K or L increases since $\frac{\partial^2 Y}{\partial K^2} < 0$, $\frac{\partial^2 Y}{\partial L^2} < 0$. Introduce $(k) = \alpha\left(\frac{K}{L}\right)^{\beta} = \alpha k^{\beta}$, so the differential equation becomes

$$\frac{dk}{dt} = s\alpha k^{\beta} - (\delta + n)k.$$

From $\frac{dk}{dt} = 0$, $k^* = \left(\frac{s\alpha}{\delta + n}\right)^{\frac{1}{1-\beta}}$. Substituting $k^* = \left(\frac{s\alpha}{\delta + n}\right)^{\frac{1}{1-\beta}}$ into $y = \alpha k^{\beta}$, the steady state level of per capita income is

$$\mathrm{y}^* = \mathrm{a}^{\frac{1}{1-\beta}}\left(\frac{\mathrm{s}}{\delta + \mathrm{n}}\right)^{\frac{\beta}{1-\beta}}.$$

The output per unit growth converges to n:

$$\frac{1}{Y}\frac{dY}{dt} = \frac{\beta}{k}\frac{dk}{dt} + n \to n.$$

A multiplicative factor in the form of technological progress, $(t) = A_0 e^{gt}$, can be introduced in the production function, so that, $Y(t) = aK(t)^{\beta}(A(t)L(t))^{1-\beta}$ and $k(t) = \frac{K(t)}{A(t)L(t)}$, leading to

$$\frac{dk}{dt} = sak^{\beta} - (\delta + n + g)k.$$

The first order nonlinear differential equation has solution

$$k(t) = \left[\frac{s\alpha}{\delta + n + g} + \left(k_0^{1-\beta} - \frac{s\alpha}{\delta + n + g}\right)e^{-(\delta+n+g)(1-\beta)t}\right]^{\frac{1}{1-\beta}}.$$

This solution includes the solution to the labour growth only model, $n = 0$. The steady state is

$$k^* = \left(\frac{s\alpha}{\delta + n + g}\right)^{\frac{1}{1-\beta}}.$$

Differentiation of $\frac{dk}{dt} = sak^{\beta} - (\delta + n + g)k$ with respect to k at k^* gives $(\beta - 1)(\delta + n + g) < 0$, the equilibrium is stable. The steady state level of per capita income is

$$y^* = a^{\frac{1}{1-\beta}}\left(\frac{s}{\delta + n + g}\right)^{\frac{\beta}{1-\beta}},$$

a constant, since s, δ, n, g are all constant.

$Y(t) = \alpha K^{\beta}\left(A_0 L_0 e^{\left(\frac{g}{1-\beta}+n\right)t}\right)^{1-\beta} = ak^{\beta}A_0 L_0 e^{\left(\frac{g}{1-\beta}+n\right)t}$. The output per unit growth, $\frac{1}{Y}\frac{dY}{dt}$, converges to $\frac{g}{1-\beta} + n$.

The Solow residual is the part of growth unexplained by changes in capital and labour. For $Y(t) = aK(t)^{\beta}(A(t)L(t))^{1-\beta}$

$$\frac{\partial Y}{\partial t} = a\beta K(t)^{\beta-1}(A(t)L(t))^{1-\beta}\frac{dK}{dt} + aK(t)^{\beta}(1-\beta)(A(t)L(t))^{-\beta}\left[\frac{dA}{dt}L(t) + \frac{dL}{dt}A(t)\right].$$

The growth rate per unit output is

$$\frac{1}{Y}\frac{\partial Y}{\partial t} = \frac{\beta}{K}\frac{dK}{dt} + (1-\beta)\frac{1}{L}\frac{dL}{dt} + (1-\beta)\frac{1}{A}\frac{dA}{dt},$$

$$\textit{Solow residual} = \frac{1}{Y}\frac{\partial Y}{\partial t} - \left[\frac{\beta}{K}\frac{dK}{dt} + (1-\beta)\frac{1}{L}\frac{dL}{dt}\right].$$

A positive Solow residual would indicate a faster output growth than that of capital and labour.

2.4 Phelps

Phelps [11] used the neoclassical growth model to address the consumption per unit of labour at equilibrium in the so-called "golden rule". At equilibrium with labour force growth rate, n, only the consumption per unit of labour is

$$c(t) = f(k) - nk.$$

For a maximum consumption per unit of labour

$$\frac{dc}{dk} = \frac{df}{dk} - n = 0.$$

Since $\frac{d^2f}{dk^2} < 0$, the turning point is a maximum given by $\frac{df}{dk} = n$. The "golden rule" concludes that the marginal output per worker must equal the growth rate of the labour force at maximum per capita consumption.

2.5 RCK

The Ramsey–Cass–Koopmans model, or RCK model, is a neoclassical model of economic growth which differs from Solow's model in its inclusion of consumption, based primarily on the work of Ramsey [12], with later significant extensions by Cass [13] and Koopmans [14].

$$\frac{dk}{dt} = f(k) - (\delta + n)k - c(t).$$

A steady state is when $c(t) = f(k) - (\delta + n)k$.

There is a second equation of the RCK model, the social planner's problem of maximizing a social welfare function expressed by the integral

$$\int_0^\infty e^{-\rho t} L(t) u(c(t)) dt = \int_0^\infty e^{(n-\rho)t} u(c(t)) dt,$$

where $\rho > 0$ is the discount rate and $u(c(t))$ is a strictly increasing concave utility function of consumption. The objective is formally stated thus

$$\mathrm{u}^* = \max_{\mathrm{c(t)}} \int_0^\infty \mathrm{e}^{(\mathrm{n}-\rho)\mathrm{t}} \mathrm{u}(\mathrm{c}(\mathrm{t})) \mathrm{dt}$$

subject to

$$\frac{\mathrm{dk}}{\mathrm{dt}} = \mathrm{f(k)} - (\delta + \mathrm{n})\mathrm{k} - \mathrm{c(t)}$$
$$k_0 = k(0).$$

The Hamiltonian is

$$\mathcal{H}(c) = e^{(n-\rho)t}\left[u(c) + \lambda e^{(\rho-n)t}(f(k) - (\delta + n)k - c(t))\right],$$

where λ is the costate variable (Lagrange multiplier). From

$$\frac{\partial \mathcal{H}}{\partial c} = e^{(n-\rho)t}\frac{\partial u}{\partial c} - \lambda = 0,$$
$$\lambda = e^{(n-\rho)t}\frac{\partial u}{\partial c}.$$

Also for the costate variable

$$\frac{d\lambda}{dt} = -\frac{\partial \mathcal{H}}{\partial k} = -\lambda\left[\frac{\partial f}{\partial k} - (\delta + n)\right],$$

and

$$\frac{d\lambda}{dt} = (n-\rho)\lambda + \frac{\frac{\partial^2 u}{\partial c^2}}{\frac{\partial u}{\partial c}}\frac{dc}{dt}\lambda.$$

Hence

$$(n-\rho) + \frac{\frac{\partial^2 u}{\partial c^2}}{\frac{\partial u}{\partial c}}\frac{dc}{dt} = -\frac{\partial f}{\partial k} + (\delta + n),$$

whence

$$\frac{dc}{dt} = \frac{\frac{\partial u}{\partial c}}{\frac{\partial^2 u}{\partial c^2}}\left[-\frac{\partial f}{\partial k} + \delta + \rho\right].$$

This is a nonlinear differential equation that describes the optimal evolution of consumption, known as the Keynes-Ramsey rule. Along with the differential equation, $\frac{dk}{dt} = f(k) - (\delta+n)k - c(t)$, form the RCK dynamical system which does not admit an analytical solution. At equilibrium,

$$\left(\frac{\partial f}{\partial k}\right)_{k^*} = \delta + \rho,$$

$$c^* = f(k^*) - (\delta+n)k^*.$$

The Jacobian matrix at equilibrium,

$$J = \begin{bmatrix} \rho - n & -1 \\ -\dfrac{\frac{\partial u}{\partial c}}{\frac{\partial^2 u}{\partial c^2}}\left(\dfrac{\partial^2 f}{\partial k^2}\right)_{k^*} & 0 \end{bmatrix}$$

has eigenvalues real and opposite in sign as its determinant is $-\frac{\frac{\partial u}{\partial c}}{\frac{\partial^2 u}{\partial c^2}}\left(\frac{\partial^2 f}{\partial k^2}\right)_{k^*} < 0$ ($f(k)$ and $u(c)$ are both concave), therefore the equilibrium is a saddle point.

2.6 Romer

The growth in the Solow model is exogenous, the steady state depends on the exogenous parameters, $,g$, which are due to outside trends. In the absence of $A(t)L(t)$ growth cannot be maintained. The marginal product of capital, $\frac{\partial Y}{\partial K} = a\beta A(t)^{1-\beta}\left(\frac{L}{K}\right)^{1-\beta} = \frac{a\beta A(t)^{1-\beta}}{\left(\frac{K}{L}\right)^{1-\beta}}$, is inversely proportional to the capital per labour, $\frac{K}{L}$. In countries with lower capital per labour the marginal product of capital should be higher which is not the case. The disparity could be attributed to the different g values in $A(t)$, which is treated as an exogenously given parameter in the Solow model, so an explanation is lacking.

Romer [15] proposed a mathematical theory of endogenous growth based on the following three assumptions:

i. The production function, $Y = F(K, A, L)$ offers increasing returns to scale, that is $F(\lambda K, \lambda A, \lambda L) > \lambda F(K, A, L)$.

ii. The change in capital is identical to Solow's model, $\frac{dK}{dt} = sY - \delta K$, where s is the fraction in savings, δ is the exogenous capital depreciation rate. Labour, L, is also exogenous, $\frac{dL}{dt} = nL$, and is comprises labour involved in research technology, L_A, and labour involved in the production of the final goods, $L_Y, L = L_A + L_Y$.

iii. Technology is exogenous and evolves in time, $\frac{dA}{dt} = \gamma L_A^{\theta} A^{\varphi}, 0 < \theta < 1, \varphi < 1$.

As is evident from the three assumptions, Romer's growth model consists of three sectors: the research sector of ideas, the intermediate goods sector which implements the ideas of the research sector and the final goods sector which produces the final output.

Let g_A be the technology growth rate, taken to be constant along the stable path,

$$g_A = \frac{1}{A}\frac{dA}{dt} = \gamma L_A^{\theta} A^{\varphi-1},$$

$$\frac{dg_A}{dt} = \gamma \theta L_A^{\theta-1} \frac{dL_A}{dt} A^{\varphi-1} + \gamma(\varphi - 1) L_A^{\theta} A^{\varphi-2} \frac{dA}{dt} = 0,$$

$$\theta \frac{1}{L_A}\frac{dL_A}{dt} + (\varphi - 1)\frac{1}{A}\frac{dA}{dt} = 0,$$

$$\theta n + (\varphi - 1) g_A = 0,$$

$$g_A = \frac{\theta n}{1 - \varphi}.$$

In Romer's model, the output production function is given by

$$y = k^{\beta}\left(\frac{L_Y}{L}\right)^{1-\beta},$$

and the capital dynamics is

$$\frac{dk}{dt} = sk^{\beta}\left(\frac{L_Y}{L}\right)^{1-\beta} - (n + g_A + \delta)k.$$

The respective stable equilibria are

$$k^* = \frac{L_Y}{L}\left(\frac{s}{n + g_A + \delta}\right)^{\frac{1}{1-\beta}},$$

$$y^* = \frac{L_Y}{L}\left(\frac{s}{n + g_A + \delta}\right)^{\frac{\beta}{1-\beta}}.$$

The labour involved in the production of the final goods, L_Y, is determined in Romer [15] by maximizing the net profit for the final goods sector and obtaining the closed form expression for $\frac{L_Y}{L} = \frac{r-n}{r-n+\beta g_A}$, where r is the interest rate, and all parameters are exogenous except for g_A which is derived endogenously.

A nice accessible exposition of both Solow's and Romer's growth models is Chu [16]. Jones [17] argued that the predicted scale effects of Romer's theory of growth is inconsistent with the time-series evidence from industrialized economies and that long-term growth depends on exogenous parameters including the rate of population growth.

2.7 Mankiw, Romer and Weil

Mankiw, Romer and Weil [18] argued that the marginal product of capital, $\frac{\partial Y}{\partial K}$, is lower in poorer countries is due to their deficiency in human capital. Human capital is the accumulation of knowledge and skills achieved through training and education, which are essential ingredients in adding economic value. The production function is of the Cobb–Douglas type

$$Y(t) = H(t)^{\alpha} K(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta} = \left(\frac{H(t)}{A(t)L(t)}\right)^{\alpha} \left(\frac{K(t)}{A(t)L(t)}\right)^{\beta} A(t)L(t),$$

$$y(t) = \frac{Y(t)}{A(t)L(t)} = \left(\frac{H(t)}{A(t)L(t)}\right)^{\alpha} \left(\frac{K(t)}{A(t)L(t)}\right)^{\beta} = h^{\alpha} k^{\beta},$$

where $H(t)$ is the human capital stock which depreciates at the same rate, δ, as $K(t)$. As in Solow's model, a fraction of the output, $sY(t)$, is saved but in this model, it is split between human and capital stock, $s = s_H + s_K$. The evolution of the economy is determined by

$$\frac{dk}{dt} = s_K h^{\alpha} k^{\beta} - (n + g + \delta) k,$$

$$\frac{dh}{dt} = s_H h^{\alpha} k^{\beta} - (n + g + \delta) h.$$

The equilibrium is

$$k^* = \left(\frac{n + g + \delta}{s_K^{1-\alpha} s_H^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}},$$

$$h^* = \left(\frac{n + g + \delta}{s_K^{\beta} s_H^{1-\beta}}\right)^{\frac{1}{\alpha+\beta-1}}.$$

In the steady state,

$$y^* = (n + g + \delta)^{\frac{\alpha+\beta}{\alpha+\beta-1}} s_K^{\frac{-\beta}{\alpha+\beta-1}} s_H^{\frac{-\alpha}{\alpha+\beta-1}}.$$

Introduce the transformations, $x_1 = \frac{k}{k^*}, x_2 = \frac{h}{h^*}$, so that the equilibrium shifts to $(1, 1)$. Then

$$\frac{dx_1}{dt} = (n + g + \delta)\left(x_1^{\beta} x_2^{\alpha} - x_1\right),$$

$$\frac{dx_2}{dt} = (n + g + \delta)\left(x_1^{\beta} x_2^{\alpha} - x_2\right).$$

For small deviations, ξ_1, ξ_2, from the equilibrium the linear system

$$\frac{d\xi_1}{dt} = (n+g+\delta)[(\beta-1)\xi_1 + \alpha\xi_2],$$
$$\frac{d\xi_2}{dt} = (n+g+\delta)[\beta\xi_1 + (\alpha-1)\xi_2].$$

The eigenvalues of the Jacobian matrix,

$$(n+g+\delta)\begin{bmatrix} \beta-1 & \alpha \\ \beta & \alpha-1 \end{bmatrix},$$

are given by the roots of the quadratic

$$\lambda^2 + (2-\alpha-\beta)\lambda + (1-\alpha-\beta) = 0.$$

From the production function, $1-\alpha-\beta > 0$. Since the sum of the eigenvalues is $\alpha+\beta-2 < 0$, and the product is $1-\alpha-\beta > 0$, both roots have negative real parts and the equilibrium point is stable.

2.8 Kaldor

Kaldor [19] presented a model of the trade cycle involving non-linear investment and saving functions that shift over time in response to capital accumulation or decumulation so that the system moves from stable equilibrium to unstable equilibrium to stable equilibrium again. In Kaldor's model investment, I, and savings, S, functions are non-linear with respect to the level of activity, X, measured in terms of employment.

Kaldor used a differential equation system with general non-linear forms. Net investment, I, and savings, S, are functions of national income, Y, and capital stock, K:

$$I = I(Y,K),$$
$$S = S(Y,K),$$
$$\frac{\partial I}{\partial Y} > 0, \frac{\partial I}{\partial K} < 0, \frac{\partial S}{\partial Y} > 0, \frac{\partial S}{\partial K} < 0,$$
$$\frac{\partial I}{\partial K} < \frac{\partial S}{\partial K}.$$

Also growth in capital determines investment is given by

$$\frac{dK}{dt} = I(Y,K).$$

Since income will rise if investment is greater than savings, the dynamics of the national income is captured by the differential equation

$$\frac{dY}{dt} = \alpha[I(Y,K) - S(Y,K)], \alpha > 0.$$

The necessary and sufficient assumptions for the generation of a perpetual cyclical movement are:

i. For normal income levels,

$$\frac{\partial I}{\partial Y} > \frac{\partial S}{\partial Y}.$$

ii. For extreme income levels, either low or high,

$$\frac{\partial I}{\partial Y} < \frac{\partial S}{\partial Y}.$$

iii. At equilibrium, where $\frac{dK}{dt} = 0$, income levels are normal.

2.9 Phillips

National governments design their expenditure policies to steer the national economy towards a desired income. The theory of feedback control or servomechanisms provides the mathematical methodology of correcting deviations of the controlled variables from their target values. Feedback policies applied to economic stability were implemented by Phillips [20].

If Y is national income and D_a is the aggregate demand then for some adjustment coefficient, $a > 0$,

$$\frac{dY}{dt} = a(D_a - Y).$$

A similar differential equation holds for the actual, D_g and target government demand, D_g^* , with $b > 0$, namely,

$$\frac{dD_g}{dt} = b\left(D_g^* - D_g\right).$$

Aggregate and government demand are related by

$$D_a = mY + D_g,$$

where m is the private sector's marginal propensity to spend.
Eliminate D_a to obtain

$$\frac{dY}{dt} = a(m-1)Y + aD_g.$$

Differentiate the above to obtain

$$\frac{\mathrm{d}^2\mathrm{Y}}{\mathrm{dt}^2} = \mathrm{a}(\mathrm{m}-1)\frac{\mathrm{dY}}{\mathrm{dt}} + \mathrm{ab}\left(\mathrm{D}_\mathrm{g}^* - \mathrm{D}_\mathrm{g}\right) = \mathrm{a}(\mathrm{m}-1)\frac{\mathrm{dY}}{\mathrm{dt}} + \mathrm{abD}_\mathrm{g}^* + \mathrm{ab}(\mathrm{m}-1)\mathrm{Y} - \mathrm{b}\frac{\mathrm{dY}}{\mathrm{dt}}$$

or

$$\frac{d^2Y}{dt^2} + [b + a(1-m)]\frac{dY}{dt} + ab(1-m)Y - abD_g^* = 0.$$

Phillips' model is thus described by the linear second-order differential equation where Y is the target variable and D_g^* is the control variable. Investigated three types of feedback policy:

i. Proportional, $D_g^* = -k_P Y$, where $k_P > 0$. This policy does not prevent income reduction and induces oscillations.

ii. Derivative, $D_g^* = -k_D \frac{dY}{dt}$, where $k_D > 0$. This policy does not prevent income reduction but avoids oscillations.

iii. Integral, $D_g^* = -k_I \int_0^t Ydt$, where $k_I > 0$. This policy prevents income reduction but can induce unstable movement.

2.10 Kalecki

Kalecki [21] was the first economist to investigate the relationship between production lags and endogenous business cycles by considering a closed economic system over a short period of time without trend. $A(t)$ is the gross capital accumulation (unconsumed goods). There is a "gestation period", θ, for any investment $I(t)$. Deliveries $L(t)$ are equal to investment orders, $I(t-\theta)$ at time, $t-\theta$:

$$L(t) = I(t-\theta).$$

Any orders placed during the "gestation period", $(t-\theta, t)$, remain unfulfilled, $A(t)$ is equal to the average of investment orders $I(t)$ allocated during the period $(t-\theta, t)$:

$$A(t) = \frac{1}{\theta}\int_{t-\vartheta}^{t} I(\tau)d\tau.$$

If $K(t)$ is the capital stock, and U its physical depreciation

$$\frac{dK}{dt} = L(t) - U = I(t-\theta) - U.$$

The rate of change in investment is for some constants, $m > 0, n > 0$:

$$\frac{dI}{dt} = m\frac{dA}{dt} - n\frac{dK}{dt} = \frac{m}{\theta}[I(t) - I(t-\theta)] - n[I(t-\theta) - U].$$

Denoting the deviation of $I(t)$ from the constant demand for restoration of the depreciated industrial equipment U by $J(t) = I(t) - U$, and differentiating $J(t)$

$$\frac{\mathrm{dJ}}{\mathrm{dt}} = \frac{\mathrm{m}}{\theta}[\mathrm{J}(\mathrm{t}) - \mathrm{J}(\mathrm{t}-\theta)] - \mathrm{nJ}(\mathrm{t}-\theta)$$

or

$$\theta\frac{dJ}{dt} + (n\theta + m)J(t-\theta) - mJ(t) = 0.$$

During the interval $t \in [-\theta, 0]$ Kalecki assumed that $J(t) = 0$. A standard way to solve this differential equation with delay is to assume a solution of the form, $De^{\alpha t}$, with D and α (where α is a complex number), to be determined. The general solution of the differential equation for some constants, c_1 and c_2 is

$$J(t) = e^{bt}[c_1 \cos(\omega t) + c_2 \sin(\omega t)].$$

The sign of the real parameter, b, classifies the behavior of the model as explosive for $b > 0$, cyclical for $b = 0$, and damped for $b < 0$.

2.11 A Solow model with lags

Zak [22] considered a version of the Solow model with delay. Capital can be used τ periods later, so at time t, the capital to be put into productive use is $k(t - \tau)$. If $f(k)$ is the production function, $s \in (0, 1)$ is the constant savings rate and $\delta \in [0, 1]$ is the constant capital depreciation rate, Zak's model is

$$\frac{dk}{dt} = sf(k(t-\tau)) - \delta k(t-\tau).$$

At equilibrium,

$$sf(k^*) = \delta k^*.$$

Deviations of the form, e^t, from equilibrium are governed by

$$\frac{dk}{dt} = \left(s\frac{df}{dk} - \delta\right)e^{-\tau},$$

with characteristic equation

$$\lambda - \left(s\frac{df}{dk} - \delta\right)e^{-\lambda\tau} = 0.$$

In many cases depending on the initial conditions, the roots of the characteristic equation have real parts with opposite signs, indicating the presence of a saddle point unlike Solow's stable model. The model exhibits endogenous cycles when the roots are purely imaginary.

2.12 Goodwin

Goodwin [23] presented a nonlinear model of nonlinear business cycles with time lags between decisions to invest and the corresponding outlays. Changes at time, t, in income, $y(t)$, induce investment outlays, $O_i(t + \theta)$, at a later time, $t + \theta$. Therefore

$$O_i(t+\theta) = \varphi\left(\frac{dy}{dt}\right) = \varphi(\dot{y}).$$

Hence the nonlinear delay differential equation modeling the evolution of income is

$$\epsilon\frac{dy(t+\theta)}{dt} + (1-\alpha)y(t+\theta) = O(t) + \varphi(\dot{y}),$$

where $O(t)$ is autonomous investment outlay and ϵ, α are constants. The derivative, $\frac{d\varphi(\dot{y})}{d\dot{y}}$, measures the rate of growth in investment with relative to the income growth, termed as acceleration coefficient. Expanding the two leading terms in

Taylor series and neglecting higher order terms, Goodwin obtained the nonlinear delay differential equation

$$\varepsilon\theta\frac{d^2y}{dt^2}+[(1-\alpha)\theta+\epsilon]\frac{dy}{dt}+(1-\alpha)y(t)-\varphi(\dot{y})=O(t).$$

Goodwin assumed that $O(t)$ is constant, $O(t)=O^*$, and introduced a new variable

$$z(t)=y(t)-\frac{O^*}{1-\alpha},$$

where $\frac{O^*}{1-\alpha}$ is the income at equilibrium. The transformed differential equation is then

$$\varepsilon\theta\frac{d^2z}{dt^2}+[(1-\alpha)\theta+\epsilon]\frac{dz}{dt}-\varphi(\dot{z})+(1-\alpha)z=0.$$

The asymptotic behavior of the transformed equilibrium, $z=0$, is determined by the eigenvalue solutions of the characteristic equation

$$\varepsilon\theta\lambda^2+[(1-\alpha)\theta+\epsilon-\dot{\varphi}(0)]\lambda+(1-\alpha)=0,$$

with characteristic roots,

$$\lambda_{1,2}=\frac{\dot{\varphi}(0)-[(1-\alpha)\theta+\epsilon]\pm\sqrt{[(1-\alpha)\theta+\epsilon-\dot{\varphi}(0)]^2-4\varepsilon\theta(1-\alpha)}}{2\varepsilon\theta}.$$

Since

$$\lambda_1\lambda_2=\frac{1-\alpha}{\varepsilon\theta}>0$$

and

$$\lambda_1+\lambda_2=\frac{\dot{\varphi}(0)-[(1-\alpha)\theta+\epsilon]}{\varepsilon\theta},$$

can be either positive or negative, both eigenvalues have positive or negative real parts. So if $\dot{\varphi}(0)<(1-\alpha)\theta+\epsilon$ the deviations from equilibrium are damped oscillatory motions, but if $\dot{\varphi}(0)>(1-\alpha)\theta+\epsilon$ the system is unstable and drifts away from the locally linearized region of stability.

2.13 A brief literature survey of current research

We close this chapter by providing a very brief snapshot of the current state of the art in theories of economic growth. Most of the very recent works cited are predominantly mathematical in nature. There is an enormous literature, not touched upon here, which employs Econometrics methods, like for instance panel data regression to estimate economic growth based on explanatory variables such as income, investment, policy indicators, education and others over several decades.

In a short article Zhao [24] discusses how technology was integrated into economic growth by Romer.

Boyko et [25] use least squares linear regression to determine the values of the coefficients at which the production functions of Cobb–Douglas in Solow's growth model provide the best fit for available statistical data. Borges et al. [26] examine the dynamics of Solow's economic growth model assuming that the labour force growth rate function is a solution of a delay differential equation thereby avoiding the use of exponential growth, $L(t) = L_0 e^{nt}$, often criticized as a rather unrealistic choice. Their approach is motivated by the fact that there are delays in entering and retiring an individual from the labour force, relative to their birth date.

Zhang et al. [27] base their analysis of how the redistribution of emission quotas would impact short-run equilibrium in a specific market of interest and long-run growth on the Solow growth model with endogenous dynamics and exogenous technological shocks.

Zhang [28] develops an endogenous growth model based on modifications of both Solow's model by introducing endogenous knowledge. and Romer's by allowing knowledge to be gained from learning as well as from research.

The paper by Caraballo et al. [29] is devoted to analysis of the stability of the economy according to an extended version of Kaldor's economic growth model. They consider the role of the government's monetary and fiscal policies and we study whether or not a time delay in implementing and the fiscal policy can affect the economic stability.

Dayal [30] considers long run historical data and uses difference equation simulation to explore the Solow growth model to assess the growth changes in the recent decade.

Perez-Trujillo et al. [31] investigate the impact of improvement in accessing innovation and knowledge on economic growth and convergence among countries using an augmented Solow-Swan growth model on data from 138 countries.

Turnovsky [32] discusses contemporary aspects of stabilization policy in reference to Phillips' contributions in a lengthy paper of substantial mathematical control theory content.

Author details

Anastasios Tsoularis
Leeds Beckett University, Leeds, United Kingdom

*Address all correspondence to: a.tsoularis@leedsbeckett.ac.uk

References

[1] P.A. Samuelson. *Foundations of Economic Analysis*. Harvard University Press, Cambridge, 1947.

[2] G. Gandolfo. *Economic Dynamics*. Springer, 1997.

[3] W.A. Brock, A.G. Malliaris. *Differential equations, Stability and Chaos in Dynamic Economics*, North Holland, 1989.

[4] D.Acemoglu, *Introduction to Modern Economic Growth*, MIT press, 2009.

[5] D. Acemoglu D. Economic Growth and Development in the Undergraduate Curriculum, The Journal of Economic Education, 2013;44:2:169-177, https://doi:10.1080/00220485.2013.770344.

[6] R.F. Harrod, An Essay in Dynamic Theory, The Economic Journal, 49(193) (1939), 14–33.

[7] E.Domar, Capital Expansion, Rate of Growth, and Employment, *Econometrica*, 14(2) (1946), 14–33.

[8] P.A. Samuelson, The stability of equilibrium: Comparative statics and dynamics, Econometrica, 9(2) (1941), 97–120.

[9] R.M. Solow, A contribution to the theory of economic growth, Quarterly Journal of Economics, 70 (1956), 65–94.

[10] Ken-Ichi Inada, On a Two-Sector Model of Economic Growth: Comments and a Generalization, The Review of Economic Studies 30 (2) (1963), 119–127.

[11] E.Phelps, The Golden Rule of Accumulation: A Fable for Growth men, The American Economic Review , 51(4) (1961), 638–643.

[12] F.Ramsey, A Mathematical Theory of Saving, The Economic Journal, 38 (152) (1928), 543–559.

[13] D.Cass, Optimum growth in an aggregative model of capital accumulation, Review of Economic Studies, 32 (1965), 233–240.

[14] T. Koopmans, On the concept of optimal economic growth, in (Study Week on the) Econometric Approach to Development Planning, chap. 4, pp. 225–87, North-Holland Publishing Co., Amsterdam, 1965.

[15] P.M. Romer, Endogenous technological change, Journal of Political Economy, 98(5) (1990), S71-S102.

[16] A.C. Chu. From Solow to Romer: Teaching endogenous technological change in undergraduate economics, International Review of Economics Education, 2018;27:10-15, https://doi: 10-1016/j.iree.2018.01.006.

[17] C.I.Jones, R&D-Based Models of Economic Growth, Journal of Political Economy, 103(4) (1995), 759–784.

[18] N.G.Mankiw, D.Romer, D.N.Weil, A Contribution to the Empirics of Economic Growth, The Quarterly Journal of Economics, 107(2) (1992), 407–437.

[19] N.Kaldor, A model of the trade cycle, The Economic Journal, 50(197) (1940), 78–92.

[20] A.W. Phillips, Stabilisation policy in a closed economy, The Economic Journal, 64(254) (1954), 290–323.

[21] M.Kalecki, A Macrodynamic Theory of Business Cycles, Econometrica, 3(3) (1935), 327–344.

[22] P.J.Zak, Kaleckian Lags in General Equilibrium, Review of Political Economy, 11(3) (1999), 321–330.

[23] R.M. Goodwin, The nonlinear accelerator and the persistence of

business cycles, Econometrica, 19(1) (1951), 1–17.

[24] R. Zhao, Technology and economic growth: From Robert Solow to Paul Romer, Hum Behav & Emerg Tech. 2019;1:62–65, https://doi.org/10.1002/hbe2.116.

[25] A. A. Boyko, V. V. Kukartsev, V. S. Tynchenko, L. N. Korpacheva, N. N. Dzhioeva, A. V. Rozhkova, S. V. Aponasenko, Using linear regression with the least squares method to determine the parameters of the Solow model, (2020) *J. Phys.: Conf. Ser.* **1582** 012016.

[26] M.J. Borges, F.Fabião, J.Teixeira, Long Cycles Versus Time Delays in a Modified Solow Growth Model. In: Bilgin M.H., Danis H., Demir E. (eds) *Eurasian Economic Perspectives. Eurasian Studies in Business and Economics*, 14(1) (2020). Springer, Cham. https://doi.org/10.1007/978-3-030-53536-0_25.

[27] A.R.Zhang, F.Zandi, H.Kim, A Simple Macroeconomic Model of Decentralized Emission Markets Based on the Solow Growth Model, 05 May 2020, https://doi.org/10.3389/fbloc.2020.00018.

[28] W-b Zhang, Creativity, Returns to Scale and Growth by Integrating Solow, Dixit-Stiglitz, and Romer, *Eastern Journal of Economics and Finance*, 5(1) (2020), 1–16.

[29] Caraballo, T., & Silva, A. (2020). Stability analysis of a delay differential Kaldor's model with government policies. Mathematica Scandinavica, 126 (1), 117–141. https://doi.org/10.7146/math.scand.a-116243.

[30] V. Dayal (2020) Growth Data and Models. In: Quantitative Economics with R. Springer, Singapore. https://doi.org/10.1007/978-981-15-2035-8_11.

[31] M. Perez-Trujillo, M. Lacalle-Calderon, The impact of knowledge diffusion on economic growth across countries, *World Development*, 132 (2020) https://doi.org/10.1016/j.worlddev.2020.104995.

[32] S.J. Turnovsky, Stabilization theory and policy: 50 years after the Phillips curve, Economica, 78 (2011), 67–78.