Qualitative Spatial Reasoning for Orientation Relations in a 3-D Context

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Abstract. Our previous work focuses on how the nine tiles in the 2-D projection-based model for cardinal directions can be partitioned into sets based on horizontal and vertical constraints (called Horizontal and Vertical Constraints Model). In this paper, the 2-D Horizontal and Vertical Constraints model is adapted and extended into a 3-D Horizontal and Vertical Constraints Block model so that it facilitates easy reasoning with 3-D volumetric regions (i.e. without holes and single-pieced) in the real physical world (e.g. intelligent robotics, building construction, etc…). This model partitions a 3-D Euclidean space of a 3-D reference region into 9 blocks, namely, *left, middle*$_x$*, right, above, middle*$_y$*, below, left, middle*$_z$*, right*. The additional central block (or the Minimum Bounding Box of the 3-D reference region) is an intersection of the three blocks, namely, *middle*$_x$*, middle*$_y$, and *middle*$_z$. The added value of the 3-D Horizontal and Vertical Constraints Block model is the use of intuitive (i.e. commonsense) knowledge representation for 3-D orientation relations. However, the underlying formal representation of the model is facilitated through the use of the 3-D Cartesian Coordinate system, first order logic, and boolean algebraic expressions. The novel contribution of this research work is fostering reasoning with partial orientation relation related knowledge (note: these are called weak relations) and also integrating mereology into the 3-D model in order to render the representation of the model more expressive. Finally, composition of relations is the technique employed in this research to general new knowledge. Mereology is integrated into the model in order to render the model more expressively. Finally, several examples will demonstrate how the model could be used to make inferences about 3-D orientation relations. AQ1

Keywords: Orientation · composition table · reasoning · mereology qualitative spatial reasoning

1 Introduction

A lot of work has been done on 2-D orientation relations. Orientation directions are generally used to describe relative positions of objects in large-scale spaces, particularly, in the geography domain. These relations specify the direction from one region to another in terms of the familiar compass bearings: north, south, east and west; or typical intuitive orientation relations (front, back, left, and right). Intermediate directions such as {north-west, north-east, south-west, and south-east} or {left-front, left-back, right-front,

K. Arai et al. (Eds.): SAI 2018, AISC 858, pp. 1–29, 2019.
https://doi.org/10.1007/978-3-030-01174-1_74

and right-back} are also often used. Classical reasoning models for orientation direction relations are the cone-shaped [7], projection-based models [7, 8] which form the basis of the 2-D Horizontal and Vertical Constraints Model [13, 14, 15], and direction matrix [9].

Vieu and colleagues [26] have built a logical framework for space. Some research has been conducted on spatial relations between 3-D regions [e.g. 17]. 3-D reasoning models have been applied to different context: building construction [6]; volumetric reasoning about objects and surfaces [16] and scene understanding [12], etc. Existing research on 3-D orientation models is still limited. Current 3-D orientation models which extend Freksa and Zimmerman [8] orientation models are: coarse 3-D orientation model [21]; 3-D cardinal direction models [4, 5, 24, 27]. However, little work has been conducted on weak orientation relations that facilitate reasoning with partial knowledge. Consequently, in this paper, the original 2-D Horizontal and Vertical Constraints model is adapted and extended into a block model.

Papadias and Theorodis [22] describe topological and direction relations between regions using their minimum bounding rectangles (MBRs). However, the language used is not sufficiently expressive to describe direction relations. Additionally, the MBR technique yields erroneous outcome when involving regions that are not rectangular in shape [9]. Some work has been done on hybrid direction models. Liu and colleagues [20] have developed reasoning algorithms which combine RCC-8 [23] for topological relations and the Cardinal Direction Calculus (CDC, [9]) for direction relations. Li and colleague's work [18] focuses on the development and evaluation of an efficient reasoning mechanism for RCC-8 and RA (Rectangle Algebra), and further details of RA is found in [2] which is employed to solve the satisfiability problem of these two joint constraint networks. Zhang and colleagues [28] develop a cubic algorithm for reasoning with cardinal directions. A review of existing work reveals that there is limited work on the 3-D hybrid direction relations. This is essential in order to foster more expressive description of a physical environment involving 3-D orientation relations. Consequently, in this paper, it is demonstrated how mereology [10] could be incorporated into a block model in order to facilitate a more expressive description of 3-D orientation relations between regions.

Typically, composition tables are used to infer spatial relations between objects. One of the advantages of composition tables is that they can lead to tractable computation of inferences [3]. They have been employed to make different inferences about 2-D directions relations [9, 19, 25] and 3-D direction relations [4, 5, 24, 27]. Some work has been done on the composition of hybrid models.

In summary, in this paper, it involves the partitioning of the Euclidean space of a 3-D reference region into nine primitive blocks namely: *left*, $middle_x$, *right*, *above*, $middle_y$, *below*, *left*, $middle_z$, *right*. The additional *central* block (or the Minimum Bounding Box of the 3-D reference region) is an intersection of the three blocks namely: $middle_x$, $middle_y$, and $middle_z$. The underlying formal representation of the model is facilitated through the use of first order logic and Boolean algebraic expressions. The reasoning mechanism to support reasoning with partial orientation related knowledge involving these nine weak relations will be demonstrated. Finally, mereology is integrated into this weak 3-D block orientation model in order to enhance its expressiveness and also to facilitate reasoning about 3-D orientation related phenomena.

2 Related Work

Typical 3-D direction models approximate spatial regions as a point or as a minimum bounding box (MBB). However, [24] use the term, MBV Minimum Bounding Volume (MBV) instead of MBB. Existing work shows a seamless extension of 2-D cardinal directions into 3-D space [4, 5, 11, 24]. 3-D direction models that have been developed are: Three-dimensional Cardinal Direction (TCD) model [4]; Objects Interaction Cube Matrix (OICM) model [5]. Hou, et al. [11] introduce the Block Cardinal Direction (BCD) calculus for reasoning with block cardinal direction relations between blocks in 3D space.

Every 3-D model reviewed comprises 27 cardinal direction relations. [24] represent each relation with a set of three elements, (u, v, w) where u, v, and w $\in \{1, 0, -1\}$ and the central voxel cell is labelled (0,0,0). In this representation, u refers to east or west, v is for north or south while w represents the relation above or below the central voxel cell. The Objects Interaction Cube Matrix (OICM) uses a $3 \times 3 \times 3$ matrix to represent all the possible 3-D cardinal direction relations [5]. Other reviewed 3-D cardinal direction relations-related research still uses the nine 2-D tiles for cardinal directions {N, S, E, W, O, NE, NW, SE, and SW} as the root representation. Three different horizontal sets of the nine 2-D cardinal direction tiles are employed to represent the 27 cardinal direction relations. They are the upper, middle, or lower planes [4, 5, 11, 27]. Representations of 3-D cardinal direction relations between two regions are as follows: upper plane – {N^U, S^U, E^U, W^U, O^U, NE^U, NW^U, SE^U, and SW^U}, middle plane - {N^M, S^M, E^M, W^M, O^M, NE^M, NW^M, SE^M, and SW^M}, lower plane - {N^B, S^B, E^B, W^B, O^B, NE^B, NW^B, SE^B, and SW^B} [4], [5]; upper plane – {UN, US, UE, UW, UO, UNE, UNW, USE, and USW}, middle plane - {RN, RS, RE, RW, RO, RNE, RNW, RSE, and RSW}, lower plane - {DN, DS, DE, DW, DO, DNE, DNW, DSE, and DSW} [27]; upper plane – {Eu, NEu, SEu, Wu, SWu, NWu, Su, Nu, Cu}, middle plane – {Em, NEm, SEm, Wm, SWm, NWm, Sm, Nm, Cm}, lower plane – {Ed, NEd, SEd, Wd, SWd, NWd, Sd, Nd, Cd} [11]. Finally, the number of compositions for the 27 binary cardinal direction relations is 27×27 which is equivalent to 729 relations.

In conclusion, all the work reviewed is confined to cardinal direction relations for geographical information systems. This paper explores the use of intuitive representation for 3-D orientation relations. This involves the use of only 9 blocks and Boolean algebra (see Table 4) to represent all the 27 direction relations represented in the models reviewed. These 9 blocks are considered weak because they only represent orientation-related information along a particular axis (i.e. x-, y-, or z axis). A stronger orientation representation will require a combination of these primitives (see Tables 3 and 4). Let us assume a scene where a drone is on the left, above and front of a robot. The algebraic representation of the 3-D binary orientation is as follows: left(robot, drone) $\wedge$ above(robot, drone) $\wedge$ front(robot, drone). Such a natural-language based representation will be sufficiently more comprehensible. Additionally, the composition of these orientation relations (note: 9 x 9 relations as shown in Table 5, is exploited for the creation of new orientation relation-related knowledge).

3 Horizontal and Vertical Constrains Block Model

To reiterate, this Horizontal and Vertical Constraints Block (HVCB) model partitions a 3-D Euclidean space of a 3-D reference region into 9 blocks, namely, left, $middle_x$, right, above, $middle_y$, below, left, $middle_z$, right. The additional central block (or the Minimum Bounding Box (MBB) of the 3-D reference region) is an intersection of the three blocks namely: $middle_x$, $middle_y$, and $middle_z$, see Fig. 1). The added value of the HVCB model is the use of intuitive (i.e commonsense) knowledge representation for 3-D orientation relations. However, the underlying formal representation of the model is facilitated through the use of first order logic and Boolean algebraic expressions (see Tables 3 and 4). Other novel contributions of this research work are fostering reasoning with partial orientation relation related knowledge (note: these are called weak relations) and also integrating mereology into the model in order to render the representation of the model more expressive.

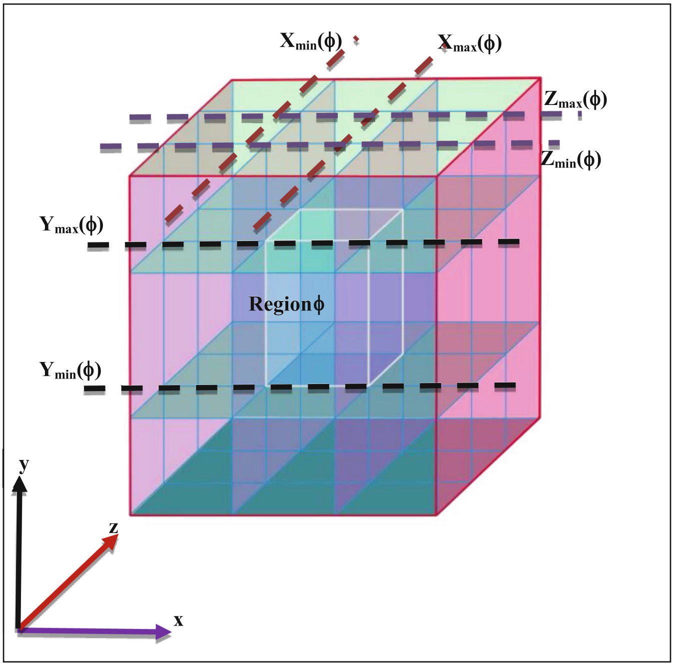

Fig. 1. 3-D horizontal and vertical constraints block model.

3.1 Definition and Formalisms for Unary Orientation Relations (in the 3-D Context)

A combined boolean algebraic method, 3-D Cartesian coordinate system, and first order logic is used to formalize the meaning of the orientation relations for an arbitrary single-pieced 3-D region without any hole (ϕ). Primitives used for this research are as follows:

- Constraints for its 3-D minimum bounding box, MBB (see Fig. 1), are as follows: x-axis - $\{X_{min}(\phi), X_{max}(\phi)\}$; y-axis - $\{Y_{min}(\phi), Y_{max}(\phi)\}$; z-axis - :$\{Z_{min}(\phi), Z_{max}(\phi)\}$. The implicit sets of constraints are as follows: $\{X_{min}(\phi) < X_{max}(\phi)\}$; $\{Y_{min}(\phi) < Y_{max}(\phi)\}$; $\{Z_{min}(\phi) < Z_{max}(\phi)\}$;
- Block, R(ϕ), is an orientation block of the extended region, ϕ. The set U = {$middle_x$(ϕ), $middle_y$(ϕ), $middle_z$(ϕ), central(ϕ), left(ϕ), right(ϕ), above(ϕ), below (ϕ), front(ϕ), back(ϕ)}.
- Set U could be further coded into subsets along the three axes: x-axis – U_x = {left (ϕ), $middle_x$(ϕ), right(ϕ)}; y-axis – U_y = {below(ϕ), $middle_y$(ϕ), above(ϕ)}; U_z = {back(ϕ), $middle_z$(ϕ), front(ϕ)}.

Definition of all the 9 weak orientation blocks in terms of the boundaries of the minimal bounding box of the extended region has been tabulated in Table 1.

Table 1. Formalism for weak unary orientation relations (in the 3-D Context)

Weak unary orientation relations	Constraints
left(ϕ)	$\{\langle x, y, z\rangle \mid x < X_{min}(\phi)\}$
$middle_x$(ϕ)	$\{\langle x, y, z\rangle \mid X_{min}(\phi) \le x \le X_{max}(\phi)\}$
right(ϕ)	$\{\langle x, y, z\rangle \mid x > X_{max}(\phi)\}$
below(ϕ)	$\{\langle x, y, z\rangle \mid y < Y_{min}(\phi)\}$
$middle_y$(ϕ)	$\{\langle x, y, z\rangle \mid Y_{min}(\phi) \le y \le Y_{max}(\phi)\}$
above(ϕ)	$\{\langle x, y, z\rangle \mid y > Y_{max}(\phi)\}$
back(ϕ)	$\{\langle x, y, z\rangle \mid z < Z_{min}(\phi)\}$
$middle_z$(ϕ)	$\{\langle x, y, z\rangle \mid Z_{min}(\phi) \le z \le Z_{max}(\phi)\}$
front(ϕ)	$\{\langle x, y, z\rangle \mid z > Z_{max}(\phi)\}$

It is noted that all the blocks in Table 1 are partially bounded except for the central block which is the result of the three intersecting middle blocks: ($middle_x$(ϕ) ∧ $middle_y$(ϕ) ∧ $middle_z$(ϕ)). The weak horizontal and vertical constraints block-related orientation representations above could be further explicated (see Tables 3 and 4) to derive more expressive orientation relations (including 27 orientation relations that are similar to the direction relations discussed in [4, 5, 11].

3.2 Definition and Formalisms for Binary Orientation Relations (in the 3-D Context)

In this paper, the 3-D HVCB binary orientation relations are coded into three categories. They are: weak 1-D binary orientation relations of the model (i.e. information of only one dimension (x-, y-, or z-axis) is known, see Table 2); weak 2-D binary orientation relations of the model (i.e. information of only two of the three dimensions (x, y, or z) are known, see Table 2); strong 3-D binary orientation relations of the model (i.e. information of all the three dimensions (x, y, and z) are known, see Table 3). Note that for all the three tables (Tables 2 to 4), region ***a*** is a reference region while ***b*** is a target region.

(1) Weak 1-D Binary Orientation Relations of the 3-D HVCB model
An exhaustive list of weak 1-D binary orientation relations is depicted in Table 2. The binary relation left(a,b) means that ***b*** is left of ***a*** where the x-coordinate ***b***, x_b, is always less than the minimum x-coordinate ($X_{min}(a)$) of ***a***'s MBB. It is considered a weak relation because it contains information of only the x-dimension (1-D) of the model. Consequently, this means that knowledge on only one-dimension (x, y, or z) is known (i.e. partial knowledge). An example of this is: in a physical multi-agent system (i.e. a fleet of drones), drone *a* merely knows that drone *b* is at its back and its representation is back($drone_a$, $drone_b$).

(2) Weak 2-D Binary Orientation Relations 3-D HVCB model
In Table 3, the binary relation left(a,b)∧above(a,b) means that ***b*** is left and above of region ***a*** where the x-coordinate of region ***b***, x_b, is always less than the minimum x-coordinate ($X_{min}(a)$) and its y-coordinate, y_b, is always greater than the maximum y-coordinate ($Y_{max}(a)$) of region ***a***'s MBB. It is considered a weak relation because it only addresses two dimensions (2-D) of the model (note: in this example, it is the x- and y- dimensions). This means that knowledge on only two of the three possible dimensions (x, y, or z) is known (i.e. partial knowledge). The representation for this weak 2-D binary orientation relation is as follows:

$R \wedge S$, $R \wedge T$, or $S \wedge T$, where $R \in U_x$, $S \in U_y$, and $T \in U_z$; based on the Commutative Law, $R \wedge S \equiv S \wedge R$

Table 2. Formalism for Weak 1-D binary orientation relations of the 3-D HVCB model

Weak binary orientation relations	Constraints
left(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a)\}$
$middle_x$(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \le x_b \le X_{max}(a)\}$
right(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a)\}$
below(a,b)	$\{\langle x, y, z\rangle \mid y_b < Y_{min}(a)\}$
$middle_y$(a,b)	$\{\langle x, y, z\rangle \mid Y_{min}(a) \le y_b \le Y_{max}(a)\}$
above(a,b)	$\{\langle x, y, z\rangle \mid y_b > Y_{max}(a)\}$
back(a,b)	$\{\langle x, y, z\rangle \mid z_b < Z_{min}(a)\}$
$middle_z$(a,b)	$\{\langle x, y, z\rangle \mid Z_{min}(a) \le z_b \le Z_{max}(a)\}$
front(a,b)	$\{\langle x, y, z\rangle \mid z_b > Z_{max}(a)\}$

Table 3. Formalism for weak 2-D binary orientation relations of the 3-D HVCB model

Weak binary orientation relations	Constraints
left(a,b)∧above(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b > Y_{max}(a)\}$
left(a,b)∧middle$_y$(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a)\}$
left(a,b)∧below(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b < Y_{min}(a)\}$
left(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge z_b < Z_{min}(a)\}$
left(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
left(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge z_b > Z_{max}(a)\}$
middle$_x$(a,b)∧above(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b > Y_{max}(a)\}$
middle$_x$(a,b)∧middle$_y$(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a)\}$
middle$_x$(a,b)∧below(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b < Y_{min}(a)\}$
middle$_x$(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge z_b < Z_{min}(a)\}$
middle$_x$(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) < x_b < X_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
middle$_x$(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge z_b > Z_{max}(a)\}$
right(a,b)∧above(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b > Y_{max}(a)\}$
right(a,b)∧middle$_y$(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a)\}$
right(a,b)∧below(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b < Y_{min}(a)\}$
right(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge z_b < Z_{min}(a)\}$
right(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
right(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge z_b > Z_{max}(a)\}$
below(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid y_b < Y_{min}(a) \wedge z_b < Z_{min}(a)\}$
below(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid y_b < Y_{min}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
below(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid y_b < Y_{min}(a) \wedge z_b > Z_{max}(a)\}$
middle$_y$(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
middle$_y$(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
middle$_y$(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
above(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid y_b > Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
above(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid y_b > Y_{max}(a)) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
above(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid y_b > Y_{max}(a) \wedge z_b > Z_{max}(a)\}$

(3) Strong 3-D Binary Orientation Relations 3-D HVCB model
In Table 4, the binary relation left(a,b)∧above(a,b) ∧back(a,b) means that ***b*** is left, above, and back of region ***a*** where the x-coordinate of region ***b***, x_b, is always less than the minimum x-coordinate ($X_{min}(a)$), its y-coordinate, y_b, is always greater than the maximum y-coordinate ($Y_{max}(a)$), and its z-coordinate, z_b, is always less than minimum z-coordinate ($Z_{min}(a)$) of region ***a***'s MBB. It is considered a strong relation because it addresses all the three dimensions (3-D) of the model (note: in this example, it is the x-, y-, and z- dimensions). This means that knowledge is considered complete. The representation for these strong 3-D binary orientations is as follows: R∧S∧T, where $R \in U_x$, $S \in U_y$, and $T \in U_z$. Once again, the Commutative Law applies to R∧S∧T.

Table 4. Expressive formalisms for strong binary orientation relations of the 3-D HVCB model

3-D binary (Block) orientation relations	Constraints
left(a,b)∧above(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b > Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
left(a,b)∧above(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b > Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
left(a,b)∧above(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b > Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
left(a,b)∧middle$_y$(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
left(a,b)∧middle$_y$(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
left(a,b)∧middle$_y$(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
left(a,b)∧below(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b < Y_{min}(a) \wedge z_b < Z_{min}(a)\}$
left(a,b)∧below(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b < Y_{min}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
left(a,b)∧below(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b < X_{min}(a) \wedge y_b < Y_{min}(a) \wedge z_b > Z_{max}(a)\}$
middle$_x$(a,b)∧above(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge yb > Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
middle$_x$ (a,b)∧above(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b > Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
middle$_x$(a,b)∧above(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b > Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
middle$_x$ (a,b)∧middle$_y$(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
middle$_x$ (a,b)∧middle$_y$(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
middle$_x$ (a,b)∧middle$_y$(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
middle$_x$(a,b)∧below(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b < Y_{min}(a) \wedge z_b < Z_{min}(a)\}$
middle$_x$ (a,b)∧below(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b < Y_{min}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
middle$_x$(a,b)∧below(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq x_b \leq X_{max}(a) \wedge y_b < Y_{min}(a) \wedge z_b > Z_{max}(a)\}$
right(a,b)∧above(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b > Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
right(a,b)∧above(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b > Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
right(a,b)∧above(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b > Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
right(a,b)∧middle$_y$(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge Y_{min}(a) \leq y \leq Y_{max}(a) \wedge z_b < Z_{min}(a)\}$
right(a,b)∧middle$_y$(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$
right(a,b)∧middle$_y$(a,b)∧front(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge Y_{min}(a) \leq y_b \leq Y_{max}(a) \wedge z_b > Z_{max}(a)\}$
right(a,b)∧below(a,b)∧back(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b < Y_{min}(a) \wedge z_b < Z_{min}(a)\}$
right(a,b)∧below(a,b)∧middle$_z$(a,b)	$\{\langle x, y, z\rangle \mid x_b > X_{max}(a) \wedge y_b < Y_{min}(a) \wedge Z_{min}(a) \leq z_b \leq Z_{max}(a)\}$

4 Composition Table

Composition is a common inference mechanism for a wide range of relations and has been exploited for automated reasoning. It is employed for reasoning about temporal descriptions of events based on intervals [1], topological relations [3, 9, 19], and direction relations [7, 13, 15, 25]. To reiterate, one of the main advantages of using composition tables is that they can lead to tractable computation of significant classes of inference.

Given the relation between *a* and *b*, the relation between *b* and *c*, a composition table allows for concluding about the relation between *a* and *c*. [3] defines the concept of the composition of two binary relations as follows.

Given a theory Θ which is used to define a set ß of mutually exhaustive and pairwise disjoint dyadic relations (i.e. a basis set). The composition, Comp(R_1,R_2), of two relations R_1, and R_2 which are taken from ß is defined to be the disjunction of all relations R_3 in ß, such that, for arbitrary constants a, b, c, the formula $R_1(a,b) \wedge R_2(b, c) \wedge R_3(a,c)$ is consistent with Θ.

4.1 Composition of Orientation Relations

The composition of two relations, R and S, is written as (R;S). It is defined by the following equivalence:

$$\forall_{p,r}[(R;S)(p,r) \leftrightarrow \exists_q[R(p,q) \wedge S(q,r)]$$

As previously mentioned, in the 3-D HVCB model, there are 9 weak orientation blocks: set U = {$\text{middle}_x(\phi)$, $\text{middle}_y(\phi)$, $\text{middle}_z(\phi)$, central(ϕ), left(ϕ), right(ϕ), above(ϕ), below(ϕ), front(ϕ), back(ϕ)}. However, the central block is derived from the intersection of three blocks namely: middle_x, middle_y, and middle_z. Thus, composition is only conducted for the other 9 weak orientation blocks (see Table 5 and these composition results will be stored in a knowledge base to) (note: due to space constraints, abbreviations are used to represent the orientation relations). Examples 1 and 2 depict how the composition results are derived.

Example 1: $L(a,b) \wedge L(b,c)$

Apply the constraints in Table 2 and it will be found that $\{\langle x, y, z\rangle | x_c < X_{min}(b) < X_{min}(a)\}$. Thus, the composition outcome ≡ L(a,c)

Example 2: $L(a,b) \wedge M_x(b,c)$

Apply the constraints in Table 2 and it will be found that $\{\langle x,y,z\rangle | X_{min}(b) < x_c < X_{max}(b) < X_{min}(a)\}$. Thus, the composition outcome ≡ L(a,c)

This paper demonstrates how composition of weak and strong 3-D orientation relations in the HVCB model could be generated based on the composition results in Table 5. They are: 1-Dx1-D (see Table 5), 1-Dx2-D and 1-Dx3-D (see Table 7); 2-Dx1-D, 2-Dx2-D, 2-Dx3-D (note: all these three sets of compositions will not be shown in this paper due to space constraints); 3-Dx1-D, 3-Dx2-D, 3-Dx3-D (note: all these sets of composition outcomes will not be shown).

Author Proof

Table 5. Composition of HVCB orientation relations (1-D X 1-D)

	L(b,c)	M_x(b,c)	R(b,c)	A(b,c)	M_y(b,c)	Be(b,c)	Ba(b,c)	M_z(b,c)	F(b,c)
L(a,b)	L(a,c)	L(a,c)	U_x(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)
M_x(a,b)	($M_x \vee$L)(a,c)	M_x(a,c)	($M_x \vee$R)(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)
R(a,b)	U_x(a,c)	R(a,c)	R(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)
A(a,b)	U(a,c)	U(a,c)	U(a,c)	A(a,c)	A(a,c)	U_y(a,c)	U(a,c)	U(a,c)	U(a,c)
M_y(a,b)	U(a,c)	U(a,c)	U(a,c)	($M_y \vee$A)(a,c)	M_y(a,c)	($M_y \vee$Be)(a,c)	U(a,c)	U(a,c)	U(a,c)
Be(a,b)	U(a,c)	U(a,c)	U(a,c)	U_y(a,c)	Be(a,c)	Be(a,c)	U(a,c)	U(a,c)	U(a,c)
Ba(a,b)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	Ba(a,c)	Ba(a,c)	U_z(a,c)
M_z(a,b)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	($M_z \vee$Ba)(a,c)	M_z(a,c)	($M_z \vee$F)(a,c)
F(a,b)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U(a,c)	U_z(a,c)	F(a,c)	F(a,c)

Note: L-left; M_x-middle$_x$, R-right; A-above, M_y-middle$_y$, Be-below, Ba-back, M_z-middle$_z$, and F-front, U_x = {L, M_x, R}, U_y = {Be, M_y, A}, U_z = {Ba, M_z, F}, U = {U_x, U_y, U_z}, $(R \vee S)(a,c) \equiv [R(a,c) \vee S(a,c)]$

Composition rules (which could be exploited as inference rules in a knowledge base) for the composition results are shown in Table 1. To reiterate: set U = {middle$_x$(ϕ), middle$_y$(ϕ), middle$_z$(ϕ), central(ϕ), left(ϕ), right(ϕ), above(ϕ), below(ϕ), front(ϕ), back(ϕ)}. In Table 6, the following is a set of inverses: {L(or R^{-1}), R (or L^{-1})}, {Be(or A^{-1}), A(or Be^{-1})}, and {Ba(or F^{-1}), F(or Ba^{-1})}.

Composition rules are as follows:

- **Composition of same orientation relation block:**

$$\forall_{p,r}[(R;R)(p,r) \leftrightarrow \exists_q[R(p,q) \wedge R(q,r)] \equiv R(p,r),\ \text{where}\ R \in U;$$

- **Composition of orientation relations in the same dimension** (x-, y-, or z-) (note: t ∈ {x, y, z}, $R_t(p,q) \in \{U_x, U_y, U_z\}$, and $M_t(p,q) \in \{M_x, M_y, M_z\}$):

$$\forall_{p,r}[(M_t; R_t)(p,r) \leftrightarrow \exists_q[M_t(p,q) \wedge R_t(q,r)] \equiv M_t(p,r) \in R_t(p,r);$$

$$\forall_{p,r}[(R_t; M_t)(p,r) \leftrightarrow \exists_q[R_t(p,q) \wedge M_t(q,r)] \equiv R_t(p,r);$$

$$\forall_{p,r}[(R_t; R_t^{-1})(p,r) \leftrightarrow \exists_q[R_t(p,q) \wedge R_t^{-1}(q,r)] \equiv U_t(p,r)$$

- **Composition of orientation relations in orthogonal dimensions** (x-axis ⊥ y-axis ⊥ z-axis) (note: t ∈ {x, y, z}, $R_t(p,q) \in \{U_x, U_y, U_z\}$, and $S_{\perp t}(p,q)$ represents orientation relations that are not orthogonal to $R_t(p,q)$)

$$\forall_{p,r}[(R_t; S_{\perp t})(p,r) \leftrightarrow \exists_q[R_t(p,q) \wedge S_{\perp t}(q,r)] \equiv U(p,r)$$

The inverse composed relations in Table 5 have been tabulated in Table 6. However, the inverse relations will be organized according to the three dimensions: x, y, and

Table 6. Inverse of the Composition of HVCB Orientation Relations (1-D X 1-D)

x-dimension $S_x(a,c)$	S_x^{-1} (a,c) or $T_x(c,a)$	y-dimension $S_y(a,c)$	S_y^{-1} (a,c) or $T_y(c,a)$	z-dimension $S_z(a,c)$	S_z^{-1} (a,c) or $T_z(c,a)$
L(a,c)	R(c,a)	A(a,c)	Be(c,a)	Ba(a,c)	F(c,a)
$M_x(a,c)$	$U_x(c,a)$	$M_y(a,c)$	$U_y(c,a)$	$M_z(a,c)$	$U_z(c,a)$
R(a,c)	L(c,a)	Be(a,c)	A(c,a)	F(a,c)	Ba(c,a)

Note: L-left; M_x-middle$_x$, R-right; A-above, M_y-middle$_y$, Be-below, Ba-back, M_z-middle$_z$, and F-front, U_x = {L, M_x, R}, U_y = {Be, M_y, A}, U_z = {Ba, M_z, F}

z. An example of application could be in a fleet of drones with one master agent while the rest are slave agents. In this context, only the master agent knows the orientation of all the slaves and not vice versa. Assume *a* as the master agent while *c*, one of the slave agents. The slave agent is known/computed to be above the master agent (i.e A(a,c)). Use Table 6, and it could be concluded that the master agent is below the slave agent (i.e. Be(c,a)).

In the previous sections, are discussion of 1-D, 2-D, and 3-D HVCB orientation relations. In this section, composition outcomes for the following are provided: 1-Dx1-D, 1-Dx2-D, 1-Dx3-D (see Table 7 for the x-dimension which is transferrable to the y- and z-dimensions); 2-Dx1-D, 2-Dx2-D, 2-Dx3-D (one example each and is tabulated in Table 8); 3-Dx1-D, 3-Dx2-D, 3-Dx3-D (one example each and is tabulated in Table 8). The outcome of the 1-Dx1-D composition has been tabulated in Table 5.

Table 7. Composition of (1-D X 2-D) and (1-D X 3-D) HVCB orientation relations

	Composition	Composition Outcome	Composition	Composition Outcome
1-D	2-D	1-D x 2-D	3-D	1-D x 3-D
L(a,b)	$L(b,c)\wedge Be(b,c)$ $L(b,c)\wedge Ba(b,c)$	L(a,c)	$L(b,c)\wedge Be(b,c)\wedge Ba(b,c)$	L(a,c)
			$L(b,c)\wedge Be(b,c)\wedge M_z(b,c)$	L(a,c)
			$L(b,c)\wedge Be(b,c)\wedge F(b,c)$	L(a,c)
	$L(b,c)\wedge M_y(b,c)$ $L(b,c)\wedge M_z(b,c)$	L(a,c)	$L(b,c)\wedge M_y(b,c)\wedge Ba(b,c)$	L(a,c)
			$L(b,c)\wedge M_y(b,c)\wedge M_z(b,c)$	L(a,c)
			$L(b,c)\wedge M_y(b,c)\wedge F(b,c)$	L(a,c)
	$L(b,c)\wedge A(b,c)$ $L(b,c)\wedge F(b,c)$	L(a,c)	$L(b,c)\wedge A(b,c)\wedge Ba(b,c)$	L(a,c)
			$L(b,c)\wedge A(b,c)\wedge M_z(b,c)$	L(a,c)
			$L(b,c)\wedge A(b,c)\wedge F(b,c)$	L(a,c)
	$M_x(b,c)\wedge Be(b,c)$ $M_x(b,c)\wedge Ba(b,c)$	L(a,c)	$M_x(b,c)\wedge Be(b,c)\wedge Ba(b,c)$	L(a,c)
			$M_x(b,c)\wedge Be(b,c)\wedge M_z(b,c)$	L(a,c)
			$M_x(b,c)\wedge Be(b,c)\wedge F(b,c)$	L(a,c)
	$M_x(b,c)\wedge M_y(b,c)$ $M_x(b,c)\wedge M_z(b,c)$	L(a,c)	$M_x(b,c)\wedge M_y(b,c)\wedge Ba(b,c)$	L(a,c)
			$M_x(b,c)\wedge M_y(b,c)\wedge M_z(b,c)$	L(a,c)
			$M_x(b,c)\wedge M_y(b,c)\wedge F(b,c)$	L(a,c)

(*continued*)

Table 7. (*continued*)

	Composition	Composition Outcome	Composition	Composition Outcome
1-D	2-D	1-D x 2-D	3-D	1-D x 3-D
	$M_x(b,c) \wedge A(b,c)$ $M_x(b,c) \wedge F(b,c)$	$L(a,c)$	$M_x(b,c) \wedge A(b,c) \wedge Ba(b,c)$	$L(a,c)$
			$M_x(b,c) \wedge A(b,c) \wedge M_z(b,c)$	$L(a,c)$
			$M_x(b,c) \wedge A(b,c) \wedge F(b,c)$	$L(a,c)$
	$R(b,c) \wedge Be(b,c)$ $R(b,c) \wedge Ba(b,c)$	$U_x(a,c)$	$R(b,c) \wedge Be(b,c) \wedge Ba(b,c)$	$U_x(a,c)$
			$R(b,c) \wedge Be(b,c) \wedge M_z(b,c)$	$U_x(a,c)$
			$R(b,c) \wedge Be(b,c) \wedge F(b,c)$	$U_x(a,c)$
	$R(b,c) \wedge M_y(b,c)$ $R(b,c) \wedge M_z(b,c)$	$U_x(a,c)$	$R(b,c) \wedge M_y(b,c) \wedge Ba(b,c)$	$U_x(a,c)$
			$R(b,c) \wedge M_y(b,c) \wedge M_z(b,c)$	$U_x(a,c)$
			$R(b,c) \wedge M_y(b,c) \wedge F(b,c)$	$U_x(a,c)$
	$R(b,c) \wedge A(b,c)$ $R(b,c) \wedge F(b,c)$	$U_x(a,c)$	$R(b,c) \wedge A(b,c) \wedge Ba(b,c)$	$U_x(a,c)$
			$R(b,c) \wedge A(b,c) \wedge M_z(b,c)$	$U_x(a,c)$
			$R(b,c) \wedge A(b,c) \wedge F(b,c)$	$U_x(a,c)$
1-D	**2-D**	**1-D x 2-D**	**3-D**	**1-D x 3-D**
$M_x(a,b)$	$L(b,c) \wedge Be(b,c)$ $L(b,c) \wedge Ba(b,c)$	$(M_x \vee L)(a,c)$	$L(b,c) \wedge Be(b,c) \wedge Ba(b,c)$	$(M_x \vee L)(a,c)$
			$L(b,c) \wedge Be(b,c) \wedge M_z(b,c)$	$(M_x \vee L)(a,c)$
			$L(b,c) \wedge Be(b,c) \wedge F(b,c)$	$(M_x \vee L)(a,c)$
	$L(b,c) \wedge M_y(b,c)$ $L(b,c) \wedge M_z(b,c)$	$(M_x \vee L)(a,c)$	$L(b,c) \wedge M_y(b,c) \wedge Ba(b,c)$	$(M_x \vee L)(a,c)$
			$L(b,c) \wedge M_y(b,c) \wedge M_z(b,c)$	$(M_x \vee L)(a,c)$
			$L(b,c) \wedge M_y(b,c) \wedge F(b,c)$	$(M_x \vee L)(a,c)$
	$L(b,c) \wedge A(b,c)$ $L(b,c) \wedge F(b,c)$	$(M_x \vee L)(a,c)$	$L(b,c) \wedge A(b,c) \wedge Ba(b,c)$	$(M_x \vee L)(a,c)$
			$L(b,c) \wedge A(b,c) \wedge M_z(b,c)$	$(M_x \vee L)(a,c)$
			$L(b,c) \wedge A(b,c) \wedge F(b,c)$	$(M_x \vee L)(a,c)$
	$M_x(b,c) \wedge Be(b,c)$ $M_x(b,c) \wedge Ba(b,c)$	$M_x(a,c)$	$M_x(b,c) \wedge Be(b,c) \wedge Ba(b,c)$	$M_x(a,c)$
			$M_x(b,c) \wedge Be(b,c) \wedge M_z(b,c)$	$M_x(a,c)$
			$M_x(b,c) \wedge Be(b,c) \wedge F(b,c)$	$M_x(a,c)$
	$M_x(b,c) \wedge M_y(b,c)$ $M_x(b,c) \wedge M_z(b,c)$	$M_x(a,c)$	$M_x(b,c) \wedge M_y(b,c) \wedge Ba(b,c)$	$M_x(a,c)$
			$M_x(b,c) \wedge M_y(b,c) \wedge M_z(b,c)$	$M_x(a,c)$
			$M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)$	$M_x(a,c)$
	$M_x(b,c) \wedge A(b,c)$ $M_x(b,c) \wedge F(b,c)$	$M_x(a,c)$	$M_x(b,c) \wedge A(b,c) \wedge Ba(b,c)$	$M_x(a,c)$
			$M_x(b,c) \wedge A(b,c) \wedge M_z(b,c)$	$M_x(a,c)$
			$M_x(b,c) \wedge A(b,c) \wedge F(b,c)$	$M_x(a,c)$
	$R(b,c) \wedge Be(b,c)$ $R(b,c) \wedge Ba(b,c)$	$(M_x \vee R)(a,c)$	$R(b,c) \wedge Be(b,c) \wedge Ba(b,c)$	$(M_x \vee R)(a,c)$
			$R(b,c) \wedge Be(b,c) \wedge M_z(b,c)$	$(M_x \vee R)(a,c)$
			$R(b,c) \wedge Be(b,c) \wedge F(b,c)$	$(M_x \vee R)(a,c)$
	$R(b,c) \wedge M_y(b,c)$ $R(b,c) \wedge M_z(b,c)$	$(M_x \vee R)(a,c)$	$R(b,c) \wedge M_y(b,c) \wedge Ba(b,c)$	$(M_x \vee R)(a,c)$
			$R(b,c) \wedge M_y(b,c) \wedge M_z(b,c)$	$(M_x \vee R)(a,c)$

(*continued*)

Table 7. (*continued*)

	Composition	Composition Outcome	Composition	Composition Outcome
1-D	2-D	1-D x 2-D	3-D	1-D x 3-D
			$R(b,c)\wedge M_y(b,c)\wedge F(b,c)$	$(M_x\vee R)(a,c)$
	$R(b,c)\wedge A(b,c)$ $R(b,c)\wedge F(b,c)$	$(M_x\vee R)(a,c)$	$R(b,c)\wedge A(b,c)\wedge Ba(b,c)$	$(M_x\vee R)(a,c)$
			$R(b,c)\wedge A(b,c)\wedge M_z(b,c)$	$(M_x\vee R)(a,c)$
			$R(b,c)\wedge A(b,c)\wedge F(b,c)$	$(M_x\vee R)(a,c)$
1-D	**2-D**	**1-D x 2-D**	**3-D**	**1-D x 3-D**
$R(a,b)$	$L(b,c)\wedge Be(b,c)$ $L(b,c)\wedge Ba(b,c)$	$U_x(a,c)$	$L(b,c)\wedge Be(b,c)\wedge Ba(b,c)$	$U_x(a,c)$
			$L(b,c)\wedge Be(b,c)\wedge M_z(b,c)$	$U_x(a,c)$
			$L(b,c)\wedge Be(b,c)\wedge F(b,c)$	$U_x(a,c)$
	$L(b,c)\wedge M_y(b,c)$ $L(b,c)\wedge M_z(b,c)$	$U_x(a,c)$	$L(b,c)\wedge M_y(b,c)\wedge Ba(b,c)$	$U_x(a,c)$
			$L(b,c)\wedge M_y(b,c)\wedge M_z(b,c)$	$U_x(a,c)$
			$L(b,c)\wedge M_y(b,c)\wedge F(b,c)$	$U_x(a,c)$
	$L(b,c)\wedge A(b,c)$ $L(b,c)\wedge F(b,c)$	$U_x(a,c)$	$L(b,c)\wedge A(b,c)\wedge Ba(b,c)$	$U_x(a,c)$
			$L(b,c)\wedge A(b,c)\wedge M_z(b,c)$	$U_x(a,c)$
			$L(b,c)\wedge A(b,c)\wedge F(b,c)$	$U_x(a,c)$
	$M_x(b,c)\wedge Be(b,c)$ $M_x(b,c)\wedge Ba(b,c)$	$R(a,c)$	$M_x(b,c)\wedge Be(b,c)\wedge Ba(b,c)$	$R(a,c)$
			$M_x(b,c)\wedge Be(b,c)\wedge M_z(b,c)$	$R(a,c)$
			$M_x(b,c)\wedge Be(b,c)\wedge F(b,c)$	$R(a,c)$
	$M_x(b,c)\wedge M_y(b,c)$ $M_x(b,c)\wedge M_z(b,c)$	$R(a,c)$	$M_x(b,c)\wedge M_y(b,c)\wedge Ba(b,c)$	$R(a,c)$
			$M_x(b,c)\wedge M_y(b,c)\wedge M_z(b,c)$	$R(a,c)$
			$M_x(b,c)\wedge M_y(b,c)\wedge F(b,c)$	$R(a,c)$
	$M_x(b,c)\wedge A(b,c)$ $M_x(b,c)\wedge F(b,c)$	$R(a,c)$	$M_x(b,c)\wedge A(b,c)\wedge Ba(b,c)$	$R(a,c)$
			$M_x(b,c)\wedge A(b,c)\wedge M_z(b,c)$	$R(a,c)$
			$M_x(b,c)\wedge A(b,c)\wedge F(b,c)$	$R(a,c)$
	$R(b,c)\wedge Be(b,c)$ $R(b,c)\wedge Ba(b,c)$	$R(a,c)$	$R(b,c)\wedge Be(b,c)\wedge Ba(b,c)$	$R(a,c)$
			$R(b,c)\wedge Be(b,c)\wedge M_z(b,c)$	$R(a,c)$
			$R(b,c)\wedge Be(b,c)\wedge F(b,c)$	$R(a,c)$
	$R(b,c)\wedge M_y(b,c)$ $R(b,c)\wedge M_z(b,c)$	$R(a,c)$	$R(b,c)\wedge M_y(b,c)\wedge Ba(b,c)$	$R(a,c)$
			$R(b,c)\wedge M_y(b,c)\wedge M_z(b,c)$	$R(a,c)$
			$R(b,c)\wedge M_y(b,c)\wedge F(b,c)$	$R(a,c)$
	$R(b,c)\wedge A(b,c)$ $R(b,c)\wedge F(b,c)$	$R(a,c)$	$R(b,c)\wedge A(b,c)\wedge Ba(b,c)$	$R(a,c)$
			$R(b,c)\wedge A(b,c)\wedge M_z(b,c)$	$R(a,c)$
			$R(b,c)\wedge A(b,c)\wedge F(b,c)$	$R(a,c)$

Next, is an example of composition for 1-Dx2-D followed by 1-Dx3-D (both of which have been tabulated in Table 7).

Example 3: $L(a,b) \wedge (L\wedge Be)(b,c)$

Apply Idempotent Law and the equivalence is:

Table 8. Examples of composition of {2-D, 3-D} X {1-D, 2-D, 3-D} HVCB orientation relations

Composition		Composition process and outcome
2-D	**1-D**	**Use Idempotent Law**
$Be(a,b) \wedge F(a,b)$	$M_x(b,c)$	$[Be(a,b) \wedge M_x(b,c)] \wedge [F(a,b) \wedge M_x(b,c)]$ $\equiv U(a,c) \wedge U(a,c) \equiv U(a,c)$
2-D	**2-D**	**Use Idempotent and Distributive Laws**
$Be(a,b) \wedge F(a,b)$	$M_x(b,c) \wedge M_y(b,c)$	$[Be(a,b) \wedge F(a,b)] \wedge [M_x(b,c) \wedge M_y(b,c)]$ $\equiv Be(a,b) \wedge [M_x(b,c) \wedge M_y(b,c)] \wedge$ $F(a,b) \wedge [M_x(b,c) \wedge M_y(b,c)]$ $\equiv [Be(a,b) \wedge M_x(b,c)] \wedge [Be(a,b) \wedge M_y(b,c)] \wedge$ $[F(a,b) \wedge M_x(b,c)] \wedge [F(a,b) \wedge M_y(b,c)] \wedge$ $\equiv U(a,c) \wedge Be(a,c) \wedge U(a,c) \wedge U(a,c)$ $\equiv Be(a,c)$
2-D	**3-D**	**Use Idempotent and Distributive Laws**
$Be(a,b) \wedge F(a,b)$	$M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)$	$[Be(a,b) \wedge F(a,b)] \wedge [M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)]$ $\equiv Be(a,b) \wedge [M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)] \wedge$ $F(a,b) \wedge [M_x(b,c) \wedge M_y(b,c)) \wedge F(b,c)]$ $\equiv [Be(a,b) \wedge M_x(b,c)] \wedge [Be(a,b) \wedge M_y(b,c)] \wedge$ $[Be(a,b) \wedge F(b,c)] \wedge$ $[F(a,b) \wedge M_x(b,c)] \wedge [F(a,b) \wedge M_y(b,c)] \wedge$ $[F(a,b) \wedge F(b,c)]$ $\equiv U(a,c) \wedge Be(a,c) \wedge U(a,c) \wedge U(a,c) \wedge U(a,c) \wedge F(a,c)$ $\equiv Be(a,c) \wedge F(a,c)$
3-D	**3-D**	**Use Idempotent and Distributive Laws** $[M_x(a,b) \wedge Be(a,b) \wedge F(a,b)]$ $\wedge [M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)]$ $\equiv M_x(a,b) \wedge [M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)] \wedge$ $Be(a,b) \wedge [M_x(b,c) \wedge M_y(b,c) \wedge F(b,c)] \wedge$ $F(a,b) \wedge [M_x(b,c) \wedge M_y(b,c)) \wedge F(b,c)]$ $\equiv [M_x(a,b) \wedge M_x(b,c)] \wedge [M_x(a,b) \wedge M_y(b,c)] \wedge$ $[M_x(a,b) \wedge F(b,c)] \wedge$ $[Be(a,b) \wedge M_x(b,c)] \wedge [Be(a,b) \wedge M_y(b,c)] \wedge$ $[Be(a,b) \wedge F(b,c)] \wedge$ $[F(a,b) \wedge M_x(b,c)] \wedge [F(a,b) \wedge M_y(b,c)] \wedge$ $[F(a,b) \wedge F(b,c)]$ $\equiv M_x(a,c) \wedge U(a,c) \wedge U(a,c) \wedge$ $U(a,c) \wedge Be(a,c) \wedge U(a,c) \wedge$ $U(a,c) \wedge U(a,c) \wedge F(a,c)$ $\equiv M_x(a,c) \wedge Be(a,c) \wedge F(a,c)$

[L(a,b) ∧ L(b,c)] ∧ [L(a,b) ∧ Be(b,c)] (note: refer to Table 5 for the composition outcome of each component)

≡ L(a,c) ∧ U(a,c) ≡ L(a,c)

Example 4: L(a,b) ∧ [L(b,c)∧Be(b,c)∧Ba(b,c)]

Apply Commutative Law and the equivalence is:

[L(a,b) ∧ L(b,c)] ∧ [L(a,b) ∧ Be(b,c)] ∧ [L(a,b) ∧ Ba(b,c)]

≡ L(a,c) ∧ U(a,c) ∧ U(a,c) ≡ L(a,c)

The same set of composition rules for 1-Dx1-D composition applies to the composition of 2-D and 3-D HVCB orientation relations.

5 Hybrid Mereology, Horizontal and Vertical Constraints Block Model

In [14, 15], a formula (obtained through case analyses) have been introduced for computing the composition of "whole" and "part" cardinal direction relations. These are adapted for the "whole" and "part" HVCB binary orientations. Assume the entire x-dimension (i.e "whole") of a target region ***b*** is in the left block of reference region, ***a***, then the representation is as follows: A_x[left(a,b)]. However, if the x-dimension of the target region ***b*** is in the left and middle$_x$ blocks of region ***a***, then the representation is as follows: P_x[left(a,b)]∧P_x[middle$_x$(a,b)].

Table 9. Formal definitions of "Whole" HVCB orientation relations

"Whole" binary orientation relations	Constraints
A_x[left(a,b)]	$\{\langle x, y, z\rangle \mid X_{max}(b) < X_{min}(a)\}$
A_x[middle$_x$(a,b)]	$\{\langle x, y, z\rangle \mid X_{min}(a) \leq X_{min}(b) \wedge X_{max}(b) \leq X_{max}(a)\}$
A_x[right(a,b)]	$\{\langle x, y, z\rangle \mid X_{min}(b) > X_{max}(a)\}$
A_y[below(a,b)]	$\{\langle x, y, z\rangle \mid Y_{max}(b) < Y_{min}(a)\}$
A_y[middle$_y$(a,b)]	$\{\langle x, y, z\rangle \mid Y_{min}(a) \leq Y_{min}(b) \wedge Y_{max}(b) \leq Y_{max}(a)\}$
A_y[above(a,b)]	$\{\langle x, y, z\rangle \mid Y_{max}(b) > Y_{max}(a)\}$
A_z[back(a,b)]	$\{\langle x, y, z\rangle \mid Z_{min}(b) < Z_{min}(a)\}$
A_z[middle$_z$(a,b)]	$\{\langle x, y, z\rangle \mid Z_{min}(a) \leq Z_{min}(b) \wedge Z_{max}(b) \leq Z_{max}(a)\}$
A_z[front(a,b)]	$\{\langle x, y, z\rangle \mid Z_{max}(b) > Z_{max}(a)\}$

5.1 Definition of "Whole" and "Part" Region in a Block Model

Formal definitions of all the specific "whole" orientation relations have been tabulated in Table 9.

The general representation for the relations is as follows:

- $A_t[R(p,q)]$: the entire t-dimension of q is in R(p), where $R \in U_t$, where $t \in \{x, y, z\}$

Assume the part of the x-dimension of a target region ***b*** is in the left block of reference region, ***a***, then the representation is as follows: $P_x[left(a,b)]$. The formal definition of all the specific "part" orientation relations have been tabulated in Table 9 while the general representation for the relations are as follows:

- $P_t[R(p,q)]$: part of the t-dimension of q is in R(p), where $R \in U_t$, where $t \in \{x,y,z\}$

Based on the definitions presented in Tables 9 and 10, different combinations of "whole" and "part" orientation relations could be formulated for 2-D or 3-D of the model (see Table 11). The formal definitions will be based on the definitions in Tables 11 and X. The universal set for the "whole" and "part" relation primitives, U = $\{A_x, A_y, A_z, P_x, P_y, P_z\}$. The number of different combinations for the 2-D "whole" and

Table 10. Formal definitions of "Part" HVCB orientation relations

"Part" binary orientation relations	Constraints
$P_x[left(a,b)]$	$\{\langle x, y, z\rangle \mid X_{min}(b) < X_{min}(a) \wedge X_{max}(b) > X_{min}(a)\}$
$P_x[middle_x(a,b)]$	$\{\langle x, y, z\rangle \mid [X_{min}(b) < X_{min}(a) \wedge X_{max}(b) > X_{min}(a) \wedge X_{max}(b) \leq X_{max}(a)]$ $\vee [X_{min}(b) \geq X_{min}(a) \wedge X_{min}(b) \leq X_{max}(a) \wedge X_{max}(b) > X_{max}(a)]$ $\vee [X_{min}(b) < X_{min}(a) \wedge X_{max}(b) > X_{max}(a)\}$
$P_x[right(a,b)]$	$\{\langle x, y, z\rangle \mid X_{min}(b) < X_{max}(a) \wedge X_{max}(b) > X_{max}(a)\}$
$P_y[below(a,b)]$	$\{\langle x, y, z\rangle \mid Y_{min}(b) < Y_{min}(a) \wedge Y_{max}(b) > Y_{min}(a)\}$
$P_y[middle_y(a,b)]$	$\{\langle x, y, z\rangle \mid [Y_{min}(b) < Y_{min}(a) \wedge Y_{max}(b) > Y_{min}(a) \wedge Y_{max}(b) \leq Y_{max}(a)]$ $\vee [Y_{min}(b) \geq Y_{min}(a) \wedge Y_{min}(b) \leq Y_{max}(a) \wedge Y_{max}(b) > Y_{max}(a)]$ $\vee [Y_{min}(b) < Y_{min}(a) \wedge Y_{max}(b) > Y_{max}(a)\}$
$P_y[above(a,b)]$	$\{\langle x, y, z\rangle \mid Y_{min}(b) < Y_{max}(a) \wedge Y_{max}(b) > Y_{max}(a)\}$
$P_z[back(a,b)]$	$\{\langle x, y, z\rangle \mid Z_{min}(b) < Z_{min}(a) \wedge Z_{max}(b) > Z_{min}(a)\}$
$P_z[middle_z(a,b)]$	$\{\langle x, y, z\rangle \mid [Z_{min}(b) < Z_{min}(a) \wedge Z_{max}(b) > Z_{min}(a) \wedge Z_{max}(b) \leq Z_{max}(a)]$ $\vee [Z_{min}(b) \geq Z_{min}(a) \wedge Z_{min}(b) \leq Z_{max}(a) \wedge Z_{max}(b) > Z_{max}(a)]$ $\vee [Z_{min}(b) < Z_{min}(a) \wedge Z_{max}(b) > Z_{max}(a)\}$
$P_z[front(a,b)]$	$\{\langle x, y, z\rangle \mid Z_{min}(b) < Z_{max}(a) \wedge Z_{max}(b) > Z_{max}(a)\}$

Table 11. Combination of "Whole" and "Part" HVCB orientation relations

Combination of "whole" and "part" orientation relations	
2-D	3-D
$A_x[R_x(a,b)] \wedge A_y[R_y(a,b)]$	$A_x[R_x(a,b)] \wedge A_y[R_y(a,b)] \wedge A_z[R_z(a,b)]$
$A_x[R_x(a,b)] \wedge A_z[R_z(a,b)]$	$A_x[R_x(a,b)] \wedge A_y[R_y(a,b)] \wedge P_z[R_z(a,b)]$
$A_x[R_x(a,b)] \wedge P_y[R_y(a,b)]$	$A_x[R_x(a,b)] \wedge P_y[R_y(a,b)] \wedge A_z[R_z(a,b)]$
$A_x[R_x(a,b)] \wedge P_z[R_y(a,b)]$	$A_x[R_x(a,b)] \wedge P_y[R_y(a,b)] \wedge P_z[R_z(a,b)]$

(*continued*)

Table 11. (*continued*)

Combination of "whole" and "part" orientation relations	
2-D	3-D
$P_x[R_x(a,b)] \wedge A_y[R_y(a,b)]$	$P_x[R_x(a,b)] \wedge A_y[R_y(a,b)] \wedge A_z[R_z(a,b)]$
$P_x[R_x(a,b)] \wedge A_z[R_y(a,b)]$	$P_x[R_x(a,b)] \wedge A_y[R_y(a,b)] \wedge P_z[R_z(a,b)]$
$P_x[R_x(a,b)] \wedge P_y[R_y(a,b)]$	$P_x[R_x(a,b)] \wedge P_y[R_y(a,b)] \wedge A_z[R_z(a,b)]$
$P_x[R_x(a,b)] \wedge P_z[R_y(a,b)]$	$P_x[R_x(a,b)] \wedge P_y[R_y(a,b)] \wedge P_z[R_z(a,b)]$
$A_y[R_y(a,b)] \wedge A_z[R_z(a,b)]$	
$A_y[R_y(a,b)] \wedge P_z[R_z(a,b)]$	
$P_y[R_y(a,b)] \wedge A_z[R_z(a,b)]$	
$P_y[R_y(a,b)] \wedge P_z[R_z(a,b)]$	

"part" orientation relations are: C(6,2) (note: the following cannot co-exist – A_x and P_x, A_y and P_y, A_z and P_z) minus 3. The outcome of the total number of possible relations is 15-3 equals to 12 possible combinations. As for the 3-D "whole" and "part" relations, the total number of combinations is C(6,3) minus 12 which is equal to 8.

5.2 Composition of "Whole" and "Part" Region in a Block

The composition of two orthogonal blocks is not considered in Tables 12 and 13 because based on the results shown in Table 5 the composition outcome is always U.

Table 12. Composition of "Whole" HVCB orientation relations

Composition of "Whole" orientation relations		Outcome of composition
X-Dimension		
$A_x[L(a,b)]$	$A_x[L(b,c)]$	$A_x[L(a,c)]$
	$A_x[M_x(b,c)]$	$A_x[L(a,c)]$
	$A_x[R(b,c)]$	$A_x[U_x(a,c)] \vee P_x[L(a,c)] \wedge P_x[M_x(a,c)] \vee P_x[M_x(a,c)] \wedge P_x[R(a,c)] \vee P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]$
$A_x[M_x(a,b)]$	$A_x[L(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee P_x[L(a,c)] \wedge P_x[M_x(a,c)]$
	$A_x[M_x(b,c)]$	$A_x[M_x(a,c)]$
	$A_x[R(b,c)]$	$A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee P_x[M_x(a,c)] \wedge P_x[R(a,c)]$
$A_x[R(a,b)]$	$A_x[L(b,c)]$	$A_x[U_x(a,c)] \vee P_x[L(a,c)] \wedge P_x[M_x(a,c)] \vee P_x[M_x(a,c)] \wedge P_x[R(a,c)] \vee P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]$
	$A_x[M_x(b,c)]$	$A_x[R(a,c)]$
	$A_x[R(b,c)]$	$A_x[R(a,c)]$

(*continued*)

Table 12. (*continued*)

Composition of "Whole" orientation relations		Outcome of composition
Y-Dimension		
$A_y[Be(a,b)]$	$A_y[Be(b,c)]$	$A_y[Be(a,c)]$
	$A_y[M_y(b,c)]$	$A_y[Be(a,c)]$
	$A_y[A(b,c)]$	$A_y[U_y(a,c)] \vee P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \vee P_y[M_y(a,c)] \wedge P_y[A(a,c)] \vee P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]$
$A_y[M_y(a,b)]$	$A_y[Be(b,c)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee P_y[Be(a,c)] \wedge P_y[M_y(a,c)]$
	$A_y[M_y(b,c)]$	$A_y[M_y(a,c)]$
	$A_y[A(b,c)]$	$A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee P_y[M_y(a,c)] \wedge P_y[A(a,c)]$
$A_y[A(a,b)]$	$A_y[Be(b,c)]$	$A_y[U_y(a,c)] \vee P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \vee P_y[M_y(a,c)] \wedge P_y[A(a,c)] \vee P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]$
	$A_y[M_y(b,c)]$	$A_y[A(a,c)]$
	$A_y[A(b,c)]$	$A_y[A(a,c)]$
Z-Dimension		
$A_z[Ba(a,b)]$	$A_z[Ba(a,b)]$	$A_z[Ba(a,c)]$
	$A_z[M_z(a,b)]$	$A_z[Ba(a,c)]$
	$A_z[F(a,b)]$	$A_z[U_z(a,c)] \vee P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \vee P_z[M_z(a,c)] \wedge P_z[F(a,c)] \vee P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]$
$A_z[M_z(a,b)]$	$A_z[Ba(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]$
	$A_z[M_z(a,b)]$	$A_z[M_z(a,c)]$
	$A_z[F(a,b)]$	$A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee P_z[M_z(a,c)] \wedge P_z[F(a,c)]$
$A_z[F(a,b)]$	$A_z[Ba(a,b)]$	$A_z[U_z(a,c)] \vee P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \vee P_z[M_z(a,c)] \wedge P_z[F(a,c)] \vee P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]$
	$A_z[M_z(a,b)]$	$A_z[F(a,c)]$
	$A_z[F(a,b)]$	$A_z[F(a,c)]$

Next, it is shown how the composition outcome in Tables 7 and 8 is applied to a physical context. Let us assume that we have three geographical objects in a scene (cottage, mountain range and a lake). Their relative locations have been depicted in Fig. 2. There are three views: pan (Fig. 2(a)); front view (Fig. 2(b)); lateral or left-side view (Fig. 2(c)). The object cottage is reference object: (***a***) while its target object is the mountain range; (***b***) however, when ***b*** is a reference object, its target object is the lake; (***c***) the relative "whole" and "part" orientation relations abstracted from Fig. 2 are tabulated in Table 14 and the composition outcome is tabulated in the same table. The composition outcome shown in Table 14 is:

Table 13. Composition of horizontal and vertical constraints ‘Whole’ and ‘part block relations (3-D X 3-D)

Composition of “Whole” and “Part” orientation relations		Outcome of composition
X-Dimension		
$A_x[L(a,b)]$	$P_x[L(b,c)]$	$A_x[L(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[M_x(b,c)]$	$A_x[L(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[R(b,c)]$	$A_x[L(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$A_x[M_x(a,b)]$	$P_x[L(b,c)]$	$A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[M_x(b,c)]$	$A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[R(b,c)]$	$A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$A_x[R(a,b)]$	$P_x[L(b,c)]$	$A_x[R(a,c)] \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)] \} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[M_x(b,c)]$	$A_x[R(a,c)] \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)] \} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[R(b,c)]$	$A_x[R(a,c)] \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)] \} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$P_x[L(a,b)]$	$A_x[L(b,c)]$	$A_x[L(a,c)]$
	$A_x[M_x(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$A_x[R(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$P_x[M_x(a,b)]$	$A_x[L(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\}$
	$A_x[M_x(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$A_x[R(b,c)]$	$A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$P_x[R(a,b)]$	$A_x[L(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$A_x[M_x(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$A_x[R(b,c)]$	$A_x[R(a,c)]$

(*continued*)

Table 13. (*continued*)

Composition of "Whole" and "Part" orientation relations		Outcome of composition
$P_x[L(a,b)]$	$P_x[L(b,c)]$	$A_x[L(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[M_x(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[R(b,c)]$	$A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$P_x[M_x(a,b)]$	$P_x[L(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[M_x(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[R(b,c)]$	$A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
$P_x[R(a,b)]$	$P_x[L(b,c)]$	$A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[M_x(b,c)]$	$A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
	$P_x[R(b,c)]$	$A_x[R(a,c)] \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee \{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}$
Y-Dimension		
$A_y[Be(a,b)]$	$P_y[Be(b,c)]$	$A_y[Be(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[M_y(b,c)]$	$A_y[Be(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[A(b,c)]$	$A_y[Be(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$A_y[M_y(a,b)]$	$P_y[Be(b,c)]$	$A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[M_y(b,c)]$	$A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[A(b,c)]$	$A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$A_y[A(a,b)]$	$P_y[Be(b,c)]$	$A_y[A(a,c)] \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[M_y(b,c)]$	$A_y[A(a,c)] \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[A(b,c)]$	$A_y[A(a,c)] \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee$

(*continued*)

Table 13. (*continued*)

Composition of "Whole" and "Part" orientation relations		Outcome of composition
		$\{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$P_y[Be(a,b)]$	$A_y[Be(a,b)]$	$A_y[Be(a,c)]$
	$A_y[M_y(a,b)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$A_y[A(a,b)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$P_y[M_y(a,b)]$	$A_y[Be(a,b)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\}$
	$A_y[M_y(a,b)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$A_y[A(a,b)]$	$A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$P_y[A(a,b)]$	$A_y[Be(a,b)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$A_y[M_y(a,b)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$A_y[A(a,b)]$	$A_y[A(a,c)]$
$P_y[Be(a,b)]$	$P_y[Be(b,c)]$	$A_y[Be(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[M_y(b,c)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[A(b,c)]$	$A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$P_y[M_y(a,b)]$	$P_y[Be(b,c)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[M_y(b,c)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[A(b,c)]$	$A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
$P_y[A(a,b)]$	$P_y[Be(b,c)]$	$A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$

(*continued*)

Table 13. (*continued*)

Composition of "Whole" and "Part" orientation relations		Outcome of composition
	$P_y[M_y(b,c)]$	$A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
	$P_y[A(b,c)]$	$A_y[A(a,c)] \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee \{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}$
Z-Dimension		
$A_z[Ba(a,b)]$	$P_z[Ba(b,c)]$	$A_z[Ba(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[M_z(b,c)]$	$A_z[Ba(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[F(b,c)]$	$A_z[Ba(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
$A_z[M_z(a,b)]$	$P_z[Ba(b,c)]$	$A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[M_z(b,c)]$	$A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[F(b,c)]$	$A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
$A_z[F(a,b)]$	$P_z[Ba(b,c)]$	$A_z[F(a,c)] \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[M_z(b,c)]$	$A_z[F(a,c)] \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[F(b,c)]$	$A_z[F(a,c)] \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
$P_zBa(a,b)]$	$A_z[Ba(a,b)]$	$A_z[Ba(a,c)]$
	$A_z[M_z(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$A_z[F(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
$P_z[M_x(a,b)]$	$A_z[Ba(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\}$
	$A_z[M_z(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$A_z[F(a,b)]$	$A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\}$

(*continued*)

Table 13. (*continued*)

Composition of "Whole" and "Part" orientation relations		Outcome of composition
$P_z[F(a,b)]$	$A_z[Ba(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$A_z[M_z(a,b)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$A_z[F(a,b)]$	$A_z[F(a,c)]$
$P_zBa(a,b)]$	$P_z[Ba(b,c)]$	$A_z[Ba(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[M_z(b,c)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[F(b,c)]$	$A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
$P_z[M_x(a,b)]$	$P_z[Ba(b,c)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[M_z(b,c)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[F(b,c)]$	$A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
$P_z[F(a,b)]$	$P_z[Ba(b,c)]$	$A_z[Ba(a,c)] \vee A_z[M_z(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[M_z(b,c)]$	$A_z[M_z(a,c)] \vee A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[M_z(a,c)] \wedge P_x[F(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$
	$P_z[F(b,c)]$	$A_z[F(a,c)] \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)]\} \vee \{P_z[Ba(a,c)] \wedge P_z[M_z(a,c)] \wedge P_z[F(a,c)]\}$

Composition Rules for x, y, and z dimensions

$t = \{x, y, z\}$; $S_t \in \{L, R, Be, A, Ba, F\}$. S^{-1} is the inverse of S. $M_t \in \{M_x, M_y, M_z\}$.

Rule 1: $A_tS_t \wedge \{P_tS_t, P_tM_t\} \equiv A_tS_t \vee [P_tS_t \wedge P_tM_t] \vee [P_tS_t \wedge P_tM_t \wedge P_tS_t^{-1}]$;

Rule 2: $P_tS_t \wedge A_tS_t \equiv A_tS_t$;

Rule 3: $P_tS_t \wedge \{A_tS_t^{-1}, A_tM_t\} \equiv A_tS_t \vee A_tS_t^{-1} \vee A_tM_t \vee [P_tS_t \wedge P_tM_t] \vee [P_tS_t^{-1} \wedge P_tM_t] \vee [P_tS_t \wedge P_tM_t \wedge P_tS_t^{-1}]$;

Rule 4: $A_tM_t \wedge \{P_tS_t, P_tM_t\} \equiv A_tS_t \vee [P_tS_t \wedge P_tM_t] \vee [P_tS_t \wedge P_tM_t \wedge P_tS_t^{-1}]$;

Rule 5: $P_tS_t \wedge P_tS_t \equiv A_tS_t \vee [P_tS_t \wedge P_tM_t] \vee [P_tS_t \wedge P_tM_t \wedge P_tS_t^{-1}]$;

Rule 6: $P_tS_t \wedge P_tM_t \equiv A_tS_t \vee A_tM_t \vee [P_tS_t \wedge P_tM_t] \vee [P_tS_t \wedge P_tM_t \wedge P_tS_t^{-1}]$;

Rule 7: $\{P_tS_t \wedge P_tM_t, P_tS_t \wedge P_tS_t^{-1}\} \equiv A_tS_t \vee A_tS_t^{-1} \vee A_tM_t \vee [P_tS_t \wedge P_tM_t] \vee [P_tS_t^{-1} \wedge P_tM_t] \vee [P_tS_t \wedge P_tM_t \wedge P_tS_t^{-1}]$;

Table 14. Formal definitions of "Part" HVCB orientation relations

R(a,b)	S(b,c)	R(a,b)∧S(b,c)
x-dimension $P_x[L(a,b)] \wedge P_x[M_x(a,b)]$ $\wedge P_x[R(a,b)]$	$A_x[R(b,c)]$	$\{P_x[L(a,b)] \wedge P_x[M_x(a,b)] \wedge P_x[R(a,b)]\} \wedge$ $\{A_x[R(b,c)]\}$ $\equiv \{P_x[L(a,b)] \wedge A_x[R(b,c)]\} \wedge$ $\{P_x[M_x(a,b)] \wedge A_x[R(b,c)]\} \wedge$ $\{P_x[R(a,b)] \wedge A_x[R(b,c)]\}$ **Use Tables 11 and 12** $\equiv \{A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee$ $\{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee$ $\{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee$ $\{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}\}$ $\wedge \{A_x[L(a,c)] \vee A_x[M_x(a,c)] \vee A_x[R(a,c)] \vee$ $\{P_x[L(a,c)] \wedge P_x[M_x(a,c)]\} \vee$ $\{P_x[M_x(a,c)] \wedge P_x[R(a,c)]\} \vee$ $\{P_x[L(a,c)] \wedge P_x[M_x(a,c)] \wedge P_x[R(a,c)]\}\}$ $\wedge \{A_x[R(a,c)]\}$ $\equiv A_x[R(a,c)]$
y-dimension $P_y[Be(a,b)] \wedge P_y[M_y(a,b)]$ $\wedge P_y[A(a,b)]$	$A_y[Be(b,c)]$	$P_y[Be(a,b)] \wedge P_y[M_y(a,b)] \wedge P_y[A(a,b)]\} \wedge$ $A_y[Be(b,c)]$ $\equiv \{P_y[Be(a,b)] \wedge A_y[Be(b,c)]\} \wedge$ $\{P_y[M_y(a,b)] \wedge A_y[Be(b,c)]\} \wedge$ $\{P_y[A(a,b)] \wedge A_y[Be(b,c)]\}$ **Use Tables 11 and 12** $\equiv \{A_y[Be(a,c)]\}$ $\wedge \{A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee$ $P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\}$ $\wedge \{A_y[Be(a,c)] \vee A_y[M_y(a,c)] \vee A_y[A(a,c)] \vee$ $\{P_y[Be(a,c)] \wedge P_y[M_y(a,c)]\} \vee$ $\{P_y[M_y(a,c)] \wedge P_y[A(a,c)]\ \} \vee$ $\{P_y[Be(a,c)] \wedge P_y[M_y(a,c)] \wedge P_y[A(a,c)]\}\}$ $\equiv A_y[Be(a,c)]$

(*continued*)

Table 14. (*continued*)

R(a,b)	S(b,c)	R(a,b)∧S(b,c)
z-dimension $P_z[Ba(a,b)]\wedge P_z[M_z(a,b)]$ $\wedge P_z[F(a,b)]$	$P_z[Ba(b,c)]\wedge$ $P_z[M_z(b,c)]$	$\{P_z[Ba(a,b)]\wedge P_z[M_z(a,b)]\wedge P_z[F(a,b)]\}\wedge$ $\{P_z[Ba(b,c)]\wedge P_z[M_z(b,c)]\}$ $\equiv\{P_z[Ba(a,b)]\wedge P_z[Ba(b,c)]\wedge$ $P_z[Ba(a,b)]\wedge P_z[M_z(b,c)]\}\wedge$ $\{P_z[M_z(a,b)]\wedge P_z[Ba(a,b)]\wedge$ $P_z[M_z(b,c)]\wedge P_z[M_z(b,c)]\}\wedge$ $\{P_z[F(a,b)]\wedge P_z[Ba(b,c)]\wedge$ $P_z[F(a,b)]\wedge P_z[M_z(b,c)]\}$ **Use Tables 11 and 12** $\equiv\{A_z[Ba(a,c)]\vee\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\}$ $\vee$ $\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}\}$ $\wedge\{A_z[Ba(a,c)]\vee A_z[M_z(a,c)]\vee\{P_z[Ba(a,c)]$ $\wedge$ $P_z[M_z(a,c)]\}\vee\{P_z[M_z(a,c)]\wedge P_x[F(a,c)]\}\vee$ $\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}\}$ $\wedge\{A_z[Ba(a,c)]\vee A_z[M_z(a,c)]\vee\{P_z[Ba(a,c)]$ $\wedge$ $P_z[M_z(a,c)]\}\vee\{P_z[M_z(a,c)]\wedge P_x[F(a,c)]\}\vee$ $\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}\}$ $\wedge\{A_z[M_z(a,c)]\vee A_z[F(a,c)]\vee\{P_z[Ba(a,c)]\wedge$ $P_z[M_z(a,c)]\}\vee\{P_z[M_z(a,c)]\wedge P_x[F(a,c)]\}\vee$ $\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}\}$ $\wedge\{A_z[Ba(a,c)]\vee A_z[M_z(a,c)]\vee\{P_z[Ba(a,c)]$ $\wedge$ $P_z[M_z(a,c)]\}\vee\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge$ $P_z[F(a,c)]\}\}$ $\wedge\{A_z[M_z(a,c)]\vee A_z[F(a,c)]\vee\{P_z[Ba(a,c)]\wedge$ $P_z[M_z(a,c)]\}\vee\{P_z[M_z(a,c)]\wedge P_x[F(a,c)]\}\vee$ $\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}\}$ $\equiv\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\}\vee\{P_z[Ba(a,c)]$ $\wedge$ $P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}\}$

Note: ***a***: cottage, ***b***: mountain range, ***c***: lake

$$A_x[R(a,c)]\wedge A_y[Be(a,c)]\wedge\{P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\vee P_z[Ba(a,c)]\wedge P_z[M_z(a,c)]\wedge P_z[F(a,c)]\}.$$

This means that the "whole" x-dimension of the lake (***c***) is on the right of the cottage (***a***) AND the "whole" y-dimension of the lake is below the cottage AND "part" of the z-dimension of the lake either covers the cottage's (back and middle$_z$) blocks or its (back and middle$_z$ and front) blocks. This result is validated with the Fig. 2(c). The composition results for the x and y dimensions are correct but the outcome for the z-dimension

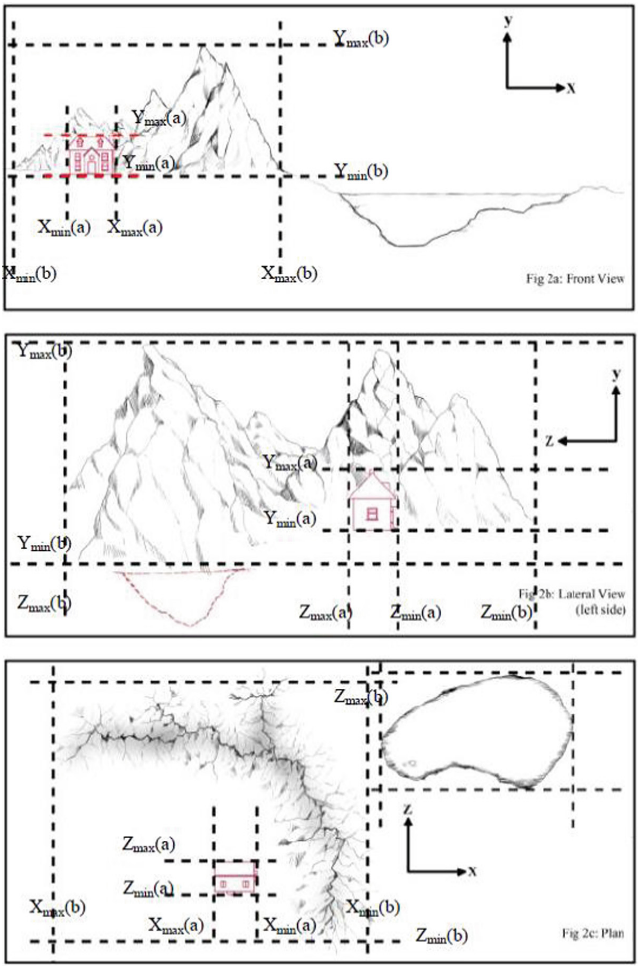

Fig. 2. A Scene with 3 different geographical objects (cottage, mountain range, and lake) and views: (a) pan, (b) front, and (c) side.

is weak because it covers all the three blocks though the correct orientation ought to be A_z[Ba(a,c)].

6 Conclusion

In this paper, a 3-D HVCB orientation model is developed based on horizontal and vertical block constraints. The model fosters reasoning with partial orientation knowledge and creation of new knowledge using a composition table. This model is integrated with mereology to render 3-D orientation relations more expressive. An application example is given to demonstrate how the 3-D mereological HVCB could be applied in a real context. Further work could be conducted to enhance reasoning with these developed "whole" and "part" 3-D HVCB orientation relations.

Acknowledgment. Deepest thanks to Stacia Low for her invaluable contribution to the diagrams.

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