An Expressive Hybrid Model for the Composition of Cardinal Directions

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Abstract

In our previous paper (Kor and Bennett, 2003), we have shown how the nine tiles in the projection-based model for cardinal directions can be partitioned into sets based on horizontal and vertical constraints (called *Horizontal and Vertical Constraints Model*). In order to come up with an expressive hybrid model for direction relations between two-dimensional single-piece regions (without holes), we integrate the well-known RCC-8 model with the above-mentioned model. From this expressive hybrid model, we derive 8 atomic binary relations and 13 feasible as well as jointly exhaustive relations for the *x* and *y* directions respectively. Based on these atomic binary relations, we derive two separate 8x8 composition tables for both the expressive and weak direction relations. We introduce a formula that can be used for the computation of the composition of expressive and weak direction relations between 'whole or part' regions. Lastly, we also show how the expressive hybrid model can be used to make several existential inferences that are not possible for existing models.

Introduction

Papadias and Theorodis (1994) describe topological and direction relations between regions using their minimum bounding rectangles (MBRs). However, the language used is not expressive enough to describe the direction relations. Additionally, the MBR technique yields erroneous outcome when the involving regions are not rectangular in shape (Goyal and Egenhofer, 2000). In order to come up with a more expressive language for direction relations, we shall combine mereological, topological, and cardinal directions relations.

Typically, composition tables are used to infer spatial relations between objects. However, existing composition tables for cardinal relations(Escrig and Toledo, 1998; Skiadopoulos and Koubarakis, 2000) are weak and not expressive enough. Consequently, these tables cannot make some existential inferences that will be shown at the later part of this paper. Here, we shall show how we come up with an expressive hybrid model for direction relations. Based on this model, we derive two 8x8 composition tables for expressive as well as weak direction relations.

In this paper, we shall describe the binary relations in the expressive hybrid model for direction relations, and define 'whole and part' relations. This is followed by introducing a formula which could be used to compute both expressive and weak direction relations for 'whole and part' regions. Finally, we shall demonstrate how the model could be used to make several types of existential inferences.

Horizontal and Vertical Constraints Model

In the projection-based model for cardinal directions (Frank, 1992), the plane of an arbitrary single-piece region *a*, is partitioned into nine tiles, North-West, NW(*a*); North, N(*a*); North-East, NE(*a*); South-West, SW(*a*); South, S(*a*); South-East, SE(*a*); West, W(*a*); Neutral Zone, O(*a*); East, E(*a*). In our previous paper (Kor and Bennett, 2003), we have shown how to partition the nine tiles into sets based on horizontal and vertical constraints called the *Horizontal and Vertical Constraints Model*. However, in this paper, we shall rename the sets for easy comprehension purposes. The following are the definitions of the partitioned regions:

- WeakNorth(*a*) is the region that covers the tiles NW(*a*), N(*a*), and NE(*a*)
- Horizontal(*a*) is the region that covers the tiles W(*a*), O(*a*), and E(*a*)
- WeakSouth(*a*) is the region that covers the tiles SW(*a*), S(*a*), and SE(*a*)
- WeakWest(*a*) is the region that covers the tiles SW(*a*), W(*a*), and NW(*a*)
- Vertical(*a*) is the region that covers the tiles S(*a*), O(*a*), and N(*a*)
- WeakEast(*a*) is the region that covers the tiles SE(*a*), E(*a*), and NE(*a*)

The set of boundaries of the minimal bounding box for region *a* is could be represented as {$X_{min}(a)$, $X_{max}(a)$, $Y_{min}(a)$, $Y_{max}(a)$}. Figure 1 depicts the horizontal and vertical sets of tiles for *a*.

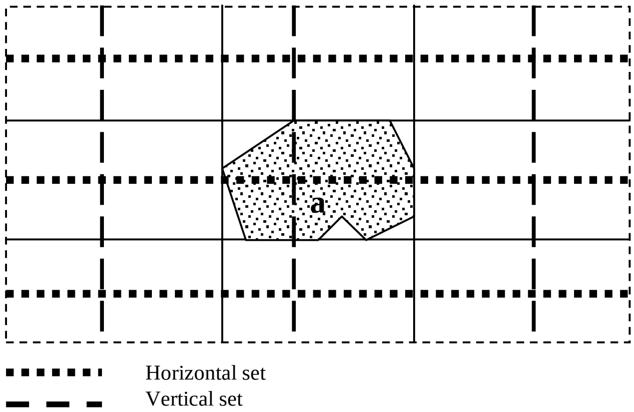


Figure 1: Horizontal and Vertical Sets of Tiles

RCC-8 Model

The RCC-8 (Randell et al, 1992) model consists of 8 atomic binary relations: PO(a,b), TPP(a,b), NTPP(a,b), EQ(a,b), NTPPi(a,b), TPPi(a,b), EC(a,b), and DC(a,b) which shall be defined later in this paper.

Expressive Hybrid Model

In order to come up with a more expressive model for the composition of cardinal directions, we integrate the *RCC-8* and the *Horizontal and Vertical Constraints Model*. We shall consider two-dimensional single-piece regions without hole. We shall consider the *x* and *y* dimensions separately.

Definitions

If there is a referent region *a*, and another arbitrary region *b*, the possible atomic binary relations between them can be defined as follows:

- *WeakNorth(b,a)* - $b \subseteq$ WeakNorth(*a*)
- *Horizontal(b,a)* - $b \subseteq$ Horizontal(*a*)
- *WeakSouth(b,a)* - $b \subseteq$ WeakSouth(*a*)
- *WeakEast(b,a)* - $b \subseteq$ WeakEast(*a*)
- *Vertical(b,a)* - $b \subseteq$ Vertical(*a*)
- *WeakWest(b,a)* - $b \subseteq$ WeakWest(*a*)
- *DCy(a,b)* – *y*-dimension of *a* is disconnected from *y*–dimension of *b*
- *EQy(a,b)* – *y*-dimension of *a* is identical with *y*–dimension of *b*
- *POy(a,b)* – *y*-dimension of *a* partially overlaps *y*–dimension of *b*
- *ECx(a,b)* – *y*-dimension of *a* is externally connected to *x*–dimension of *b*
- *TPPy(a,b)* – *y*-dimension of *a* is a tangential proper part of *y*–dimension of *b*
- *NTPPy(a,b)* – *y*-dimension of *a* is a nontangential proper part of *y*–dimension of *b*
- *TPPiy(a,b)* – *y*-dimension of *a* is a tangential proper part of *y*–dimension of *a*
- *NTPPiy(a,b)* – *y*-dimension of *b* is a nontangential proper part of *y*–dimension of *a*

Atomic Binary Relations of the Hybrid Model

In this section, we shall demonstrate how we come up with all possible binary direction relations for the hybrid model. All the possible atomic binary relations for each horizontal set are shown in Figure 2. The notations that will be used in this section are:

- *RELy(b,Z)* is any atomic binary relation between *b* and the horizontally partitioned region, Z
- *RELx(b,Z)* is any atomic binary relation between *b* and the vertically partitioned region, Z

Based on Table 1, the total number of possible binary relations for the hybrid model in the *y*-direction is [(2+4+2) + (2x4) + (2x2) + (4x2) + (2x4x2)] which equals 44 cases. However, due to the single-piece condition, the following rules apply:

- **Rule 1:** $b \subseteq \neg$(WeakNorth(*a*) $\wedge$WeakSouth(*a*))
- **Rule 2:** Assume *U* to be {WeakNorth(*a*), Horizontal(*a*), WeakSouth(*a*)}. If *NTPPy(b,R)* where $R \in U$ then $\neg$[[*NTPPy(b,R)* $\wedge$ *RELy(b,S)*] $\vee$ [*NTPPy(b,R)* $\wedge$ *RELy(b,S)* $\wedge$ *RELy(b,T)*]] where $S \in U - R$, $T \in U – S$.
- **Rule 3:** Assume *U* to be {WeakNorth(*a*), WeakSouth(*a*)}. If [*TPPy(b,Horizontal(a))* $\wedge$ *ECy(b,R)*] where $R \in U$ then $\neg$[*TPPy(b,Horizontal)* $\wedge$ *ECy(b,R)* $\wedge$ *RELy(b,S)*] where $S \in U - R$.

Based on the rules above, the total number of feasible binary relations for single-piece regions in the *y*-direction is (44 – 4 – 23 – 4) which equals 13 cases. The thirteen feasible and jointly exhaustive binary relations for the hybrid model are depicted in Figure 3. This means that in the hybrid model, the number of jointly exhaustive binary relations (in both the *x* and *y* directions) that hold between two single-piece regions will be 13x13. This concurs with the 13x13 atomic relations in the Rectangle Algebra Model (Balbiani et al. 1998).

Combined Mereological, Topological and Cardinal Direction Relations

In this section, we shall make two distinctions: 'whole and part' cardinal directions , as well as 'weak and expressive' relations. We shall rewrite the notations used in our previous paper (Kor and Bennett, 2003). $P_R(b,a)$ means that only part of the destination extended region, *b*, is in tile R(*a*). The direction relation $A_R(b,a)$ means that whole destination extended region, *b*, is in the tile R(*a*). As an example, when *b* is completely in the South-East tile of *a*, this direction relation can be represented as below:

$$A_{SE}(b,a) = \neg P_N(b,a) \wedge \neg P_{NE}(b,a) \wedge \neg P_{NW}(b,a) \wedge \neg P_S(b,a) \wedge P_{SE}(b,a) \wedge \neg P_{SW}(b,a) \wedge \neg P_W(b,a) \wedge \neg P_E(b,a) \wedge \neg P_O(b,a)$$

The 'whole and weak' direction relations are defined in terms of horizontal and vertical sets.

- $A_N(b,a) \equiv WeakNorth(b,a) \wedge Vertical(b,a)$
- $A_{NE}(b,a) \equiv WeakNorth(b,a) \wedge WeakEast(b,a)$
- $A_{NW}(b,a) \equiv WeakNorth(b,a) \wedge WeakWest(b,a)$
- $A_S(b,a) \equiv WeakSouth(b,a) \wedge Vertical(b,a)$
- $A_{SE}(b,a) \equiv WeakSouth(b,a) \wedge WeakEast(b,a)$
- $A_{SW}(b,a) \equiv WeakSouth(b,a) \wedge WeakWest(b,a)$
- $A_E(b,a) \equiv Horizontal(b,a) \wedge Vertical(b,a)$
- $A_W(b,a) \equiv Horizontal(b,a) \wedge WeakEast(b,a)$
- $A_O(b,a) \equiv Horizontalh(b,a) \wedge WeakWest(b,a)$

The 'whole and expressive' direction relations are defined in terms of expressive horizontal and vertical sets. A general form of such direction relation can be represented as follows:

$$^{RELy(b,H)}[A_R(b,a)]_{RELx(b,V)} \equiv RELy(b,H) \wedge RELx(b,V) \ \ldots(1)$$

where H and V are horizontally and vertically partitioned regions for *a* respectively, and R(*a*) $\subset$ H $\wedge$ R(*a*) $\subset$ V.

Composition Table

The composition tables computed for the horizontal and vertical sets are shown in Table 1 and Table 2. The composition of atomic binary relations in the expressive model can be collapsed into weak relations as shown in the above-mentioned tables (with shaded boxes).

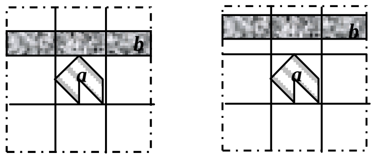
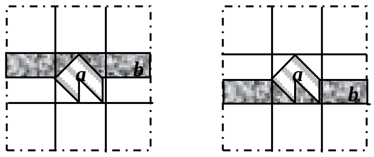
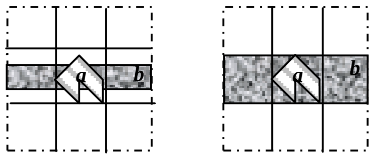
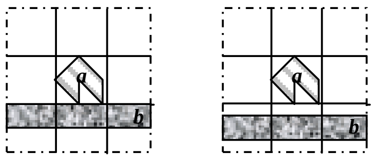
Region	Atomic binary relations
WeakNorth(a)	 TPPy(b, WeakNorth(a)) ∧ ECy(b,Horizontal(a)); NTPPy(b, WeakNorth(a))
Horizontal(a)	 TPPy(b, Horizontal(a)) ∧ ECy(b,WeakNorth(a)); TPPy(b, Horizontal(a)) ∧ ECy(b,WeakSouth(a))  NTPPy(b, Horizontal(a)); EQy(b, Horizontal(a))
WeakSouth(a)	 TPPy(b, WeakSouth(a)) ∧ ECy(b,Horizontal(a)); NTPPy(b, WeakSouth(a))

Figure 2: Possible atomic binary relations for each horizontally partitioned region

Composition of Regions with Parts

In our previous paper (Kor and Bennett, 2003), the method for the computation of part regions is not robust enough because it does not hold for all cases. In order to address such a problem, we introduce a formula (Equation 2.a). The basis of the formula is to consider the direction relation between a and each individual part of b followed by the direction relation between each individual part of b and c. Assume that the region b covers one or more than one tile of region a while region c encompasses one or more than one tile of region b. The formula for the composition of weak direction relations can be written as follows:

$$P_R(b,a) \wedge P_R(c,b)$$
$$\equiv [P_{R1}(b_1,a) \wedge P_{R2}(b,a) \ldots \wedge P_{Rk}(b_k,a)] \wedge [P_R(c,b)]$$

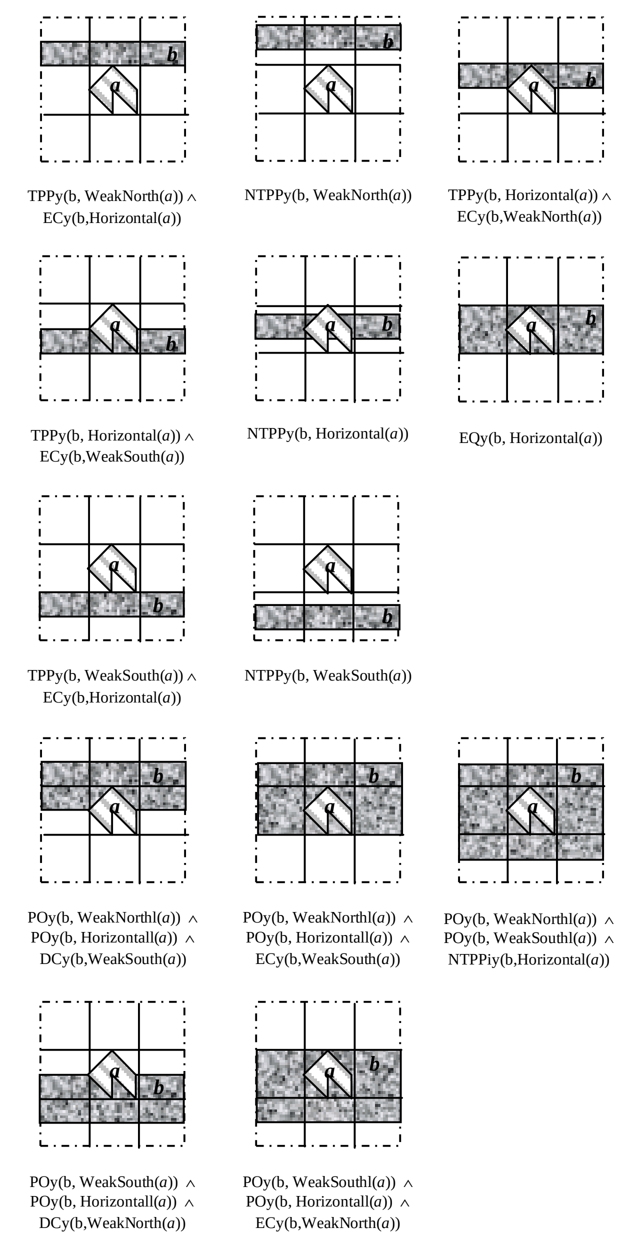


Figure 3: Thirteen feasible and jointly exhaustive binary relations in the y-direction for the hybrid model

Firstly, establish the direction relation between each individual part of b and c. The above composition can be rewritten as follows:

$$[[P_{R1}(b_1,a)] \wedge [P_{R11}(c_1,b_1) \wedge P_{R12}(c_2,b_1) \ldots \wedge P_{R1m}(c_m,b_1)]] \wedge$$
$$[[P_{R2}(b_2,a)] \wedge [P_{R21}(c_1,b_2) \wedge P_{R22}(c_2,b_2) \ldots \wedge P_{R2m}(c_m,b_2)]] \wedge$$
$$\vdots$$
$$[[P_{Rk}(b_k,a)] \wedge [P_{Rk1}(c_1,b_k) \wedge P_{Rk2}(c_2,b_k) \ldots \wedge P_{Rkm}(c_m,b_k)]] \quad \textbf{...(2)}$$

where $1 \le k \le 9$, and $1 \le m \le 9$

Consider the composition of direction for each individual part of b and c. Equation (1) becomes:

$$\big[[P_{R1}(b_1,a) \wedge P_{R11}(c_1,b_1)] \vee [P_{R1}(b_1,a) \wedge P_{R12}(c_2,b_1)] \vee \ldots [P_{R1}(b_1,a) \wedge P_{R1m}(c_m,b_1)]\big] \wedge$$
$$\big[[P_{R2}(b_2,a) \wedge P_{R21}(c_1,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R22}(c_2,b_2)] \vee \ldots [P_{R2}(b_2,a) \wedge P_{R2m}(c_m,b_2)]\big] \wedge$$
:
$$\big[[P_{Rk}(b_k,a) \wedge P_{Rk1}(c_1,b_k)] \vee [P_{Rk}(b_k,a) \wedge P_{Rk2}(c_2,b_k)] \vee \ldots [P_{Rk}(b_k,a) \wedge P_{Rkm}(c_m,b_k)]\big]$$

. . .(2.a)

By applying distributive law we have the following equation:

$$\big[[P_{R1}(b_1,a) \wedge [P_{R11}(c_1,b_1) \vee P_{R12}(c_2,b_1) \vee \ldots P_{R1m}(c_m,b_1)]\big] \wedge$$
$$\big[[P_{R2}(b_2,a) \wedge [P_{R21}(c_1,b_2) \vee P_{R22}(c_2,b_2) \vee \ldots P_{R2m}(c_m,b_2)]\big] \wedge$$
:
$$\big[[P_{Rk}(b_k,a) \wedge [P_{Rk1}(c_1,b_k) \vee P_{Rk2}(c_2,b_k) \vee \ldots P_{Rkm}(c_m,b_k)]\big]$$

. . .(2.b)

Firstly, we shall demonstrate how to apply the formula for the composition of weak direction relations followed by more expressive direction relations.

Composition of Weak Direction Relations

Type 1: $A_R(b,a) \wedge A_R(c,b)$

Find the composition of $A_O(b,a) \wedge A_{SW}(c,b)$

Use Equation **2.a** with $k = 1$, and $m = 1$.

$P_{R1}(b_1,a) \wedge P_{R11}(c_1,b_1) \equiv P_O(b,a) \wedge P_{SW}(c,b)$

$$\equiv [Horizontal(b,a) \wedge Vertical(b,a)] \wedge [WeakSouth(c,b) \wedge WeakWest(c,b)]$$
$$\equiv [Horizontal(b,a) \wedge WeakSouth(c,b)] \wedge [Vertical(b,a) \wedge WeakWest(c,b)]$$

The outcome of the composition is:

$$[Horizontal(c,a) \vee WeakSouth(c,a)] \wedge [Vertical(c,a) \vee WeakWest(c,a)]$$

This means that the region $c \subseteq$ O(a)$\vee$W(a)$\vee$S(a)$\vee$SW(a).

Type 2: $A_R(b,a) \wedge P_R(c,b)$

Find the composition of $A_E(b,a) \wedge [P_{NW}(c,b) \wedge P_N(c,b)]$

Use Equation **2.a** with $k = 1$, and $1 \leq m \leq 2$.

$$\big[[P_{R1}(b_1,a) \wedge P_{R11}(c_1,b_1)] \vee [P_{R1}(b_1,a) \wedge P_{R12}(c_2,b_1)]\big]$$
$$\equiv \big[[P_E(b,a) \wedge P_{NW}(b,c_1)] \vee [P_E(b,a) \wedge P_N(b,c_2)]\big]$$
$$\equiv \big[[Horizontal(b,a) \wedge WeakEast(b,a)] \wedge [WeakNorth(c_1,b) \wedge WeakWest(c_1,b)]\big] \vee$$
$$\big[[Horizontal(b,a) \wedge WeakEast(b,a)] \wedge [WeakNorth(c_2,b) \wedge Vertical(c_2,b)]$$
$$\equiv \big[[Horizontal(b,a) \wedge WeakNorth(c_1,b)] \wedge [WeakEast(b,a) \wedge WeakWest(c_1,b)]\big] \vee$$
$$\big[[Horizontal(b,a) \wedge WeakNorth(c_2,b)] \wedge [WeakEast(b,a) \wedge Vertical(c_2,b)]\big]$$

The outcome of the composition is:

$$\big[[Horizontal(c_1,a) \vee WeakNorth(c_1,a)] \wedge [WeakEast(c_1,a) \vee Vertical(c_1,a) \vee WeakWest(c_1,a)]\big] \vee$$
$$\big[[Horizontal(c_2,a) \vee WeakNorth(c_2,a)] \wedge [WeakEast(c_2,a)]\big]$$

Both $c_1 \subset c$ and $c_2 \subset c$, so the above outcome can be written as:

$$\big[[Horizontal(c,a) \vee WeakNorth(c,a)] \wedge [WeakEast(c,a) \vee Horizontal(c,a) \vee WeakWest(c,a)]\big]$$

This means that the region $c \subseteq$ E(a)$\vee$O(a)$\vee$W(a)$\vee$ NE(a)$\vee$N(a)$\vee$NW(a).

Type 3: $P_R(b,a) \wedge A_R(c,b)$

Find the composition of $[P_O(b_1,a) \wedge P_N(b_2,a)] \wedge A_{NE}(c,b)$

Establish the relationship between c and each individual part of b. In this case, when $A_{NE}(c,b)$, $P_{NE}(c,b_1)$ and $P_{NE}(c,b_2)$ holds (this is not necessarily true for all cases).

Use Equation **2.a** with $1 \leq k \leq 2$ and $m = 1$.

$$\big[[P_{R1}(b_1,a)] \wedge [P_{R11}(c_1,b_1)]\big] \wedge \big[[P_{R2}(b_2,a)] \wedge [P_{R21}(c_1,b_2)]\big]$$
$$\equiv \big[[P_O(b_1,a)] \wedge [P_{NE}(c,b)]\big] \wedge \big[[P_N(b_2,a)] \wedge [P_{NE}(c,b)]\big]$$

Therefore, the above composition can be rewritten as:

$$\big[[P_O(b_1,a)] \wedge [P_{NE}(c,b_1)]\big] \wedge \big[[P_N(b_2,a)] \wedge [P_{NE}(c,b_2)]\big]$$
$$\equiv \big[[Horizontal(b_1,a) \wedge Vertical(b_1,a)] \wedge [WeakNorth(c,b_1) \wedge WeakEast(c,b_1)]\big] \wedge$$
$$\big[[WeakNorth(b_2,a) \wedge Vertical(b_2,a)] \wedge [WeakNorth(c,b_2) \wedge WeakEast(c,b_2)]$$
$$\equiv \big[[Horizontal(b_1,a) \wedge WeakNorth(c,b_1)] \wedge [Vertical(b_1,a) \wedge WeakEast(c,b_1)]\big] \wedge$$
$$\big[[WeakNorth(b_2,a) \wedge WeakNorth(c,b_2)] \wedge [Vertical(b_2,a) \wedge WeakEast(c,b_2)]$$

The outcome of the composition is:

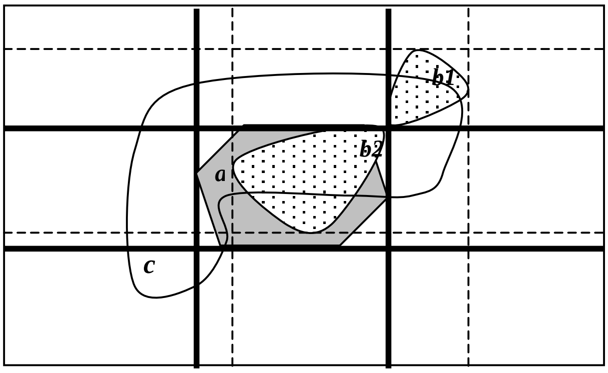
$$\big[[Horizontal(c,a) \vee WeakNorth(c,a)] \wedge [WeakEast(c,a) \vee Vertical(c,a)]\big] \wedge$$
$$\big[[\text{NTPPy(c, WeakNorth(a)}] \wedge [WeakEast(c,a) \vee Vertical(c,a)]\big]$$

$$= \big[[\text{NTPPy(c, WeakNorth(a)}] \wedge [WeakEast(c,a) \vee Vertical(c,a)]\big]$$

This means that the $Y_{min}(c)$ of the minimal bounding box for region c is greater than $Y_{max}(a)$ of the minimal bounding box for region a and $c \subseteq$ NE(a)$\vee$N(a).

Type 4: $P_R(b,a) \wedge P_R(c,b)$
Find the composition of
$[P_O(b_1,a) \wedge P_{NE}(b_2,a)] \wedge [P_O(c,b) \wedge P_W(c,b) \wedge P_{SW}(c,b)]$

The diagram Figure 4 has been drawn for this example.



Boundaries of minimal bounding box for region *a*
Boundaries of minimal bounding box for region *b*

Figure 4: An example

Establish the direction relation between each individual part of *b* and *c*.
Use Equation **2.a** with $1 \leq k \leq 2$, the value of m_1 for b_1 is $1 \leq m_1 \leq 4$. while the value m_2 for b_2 is $1 \leq m_2 \leq 7$.

$$\big[[P_{R1}(b_1,a) \wedge P_{R11}(c_1,b_1)] \vee [P_{R1}(b_1,a) \wedge P_{R12}(c_2,b_1)] \vee [P_{R1}(b_1,a) \wedge P_{R13}(c_3,b_1)] \vee [P_{R1}(b_1,a) \wedge P_{R14}(c_4,b_1)]\big] \wedge$$
$$\big[[P_{R2}(b_2,a) \wedge P_{R21}(c_1,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R22}(c_2,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R23}(c_3,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R24}(c_4,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R25}(c_5,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R26}(c_6,b_2)] \vee [P_{R2}(b_2,a) \wedge P_{R27}(c_7,b_2)]\big]$$
$$\equiv \big[[P_O(b_1,a) \wedge P_S(c_1,b_1)] \vee [P_O(b_1,a) \wedge P_{SW}(c_2,b_1)] \vee [P_O(b_1,a) \wedge P_W(c_3,b_1)] \vee [P_O(b_1,a) \wedge P_O(c_4,b_1)]\big] \wedge$$
$$\big[[P_{NE}(b_2,a) \wedge P_{NE}(c_1,b_2)] \vee [P_{NE}(b_2,a) \wedge P_N(c_2,b_2)] \vee [P_{NE}(b_2,a) \wedge P_{NW}(c_3,b_2)] \vee [P_{NE}(b_2,a) \wedge P_E(c_4,b_2)] \vee [P_{NE}(b_2,a) \wedge P_O(c_5,b_2)] \vee [P_{NE}(b_2,a) \wedge P_W(c_6,b_2)] \vee [P_{NE}(b_2,a) \wedge P_{SW}(c_7,b_2)]\big]$$

Apply the distributive law as in Equation 2.b, and we will get:

$$\big[[P_{NE}(b_1,a) \wedge [P_S(c_1,b_1) \vee P_{SW}(c_2,b_1) \vee P_W(c_3,b_1) \vee P_O(c_4,b_1)]\big] \ldots \text{part(1)}$$
$$\wedge$$
$$\big[[P_O(b_2,a) \wedge [P_{NE}(c_1,b_2) \vee P_N(c_2,b_2) \vee P_{NW}(c_3,b_2) \vee P_E(c_4,b_2) \vee P_O(c_5,b_2) \vee P_W(c_6,b_2) \vee P_{SW}(c_7,b_2)]\big] \ldots \text{part(2)}$$

In part(1) of composition, $c_1, c_2, c_3, c_4 \subset c$. To simplify the composition, we consider the combined horizontal and vertical sets of all the parts of *c*. Thus we have the following:

$$[WeakNorth(b_1,a) \wedge WeakEast(b_1,a)] \wedge$$
$$\big[[Horizontal(c,b_1) \vee WeakSouth(c,b_1)] \wedge [Vertical(c,b_1) \vee WeakWest(c,b_1)]\big]$$
$$\equiv \big[[WeakNorth(b_1,a)] \wedge [Horizontal(c,b_1) \vee WeakSouth(c,b_1)]\big] \wedge \big[[WeakEast(b_1,a)] \wedge [Vertical(c,b_1) \vee WeakWest(c,b_1)]\big]$$
$$= WeakNorth(c,a) \vee Horizontal(c,a) \vee WeakSouth(c,a)] \wedge [WeakEast(c,a) \vee Vertical(c,a) \vee WeakWest(c,a)]$$

In part(2) of composition, $c_1, c_2, c_3, c_4, c_5, c_6, c_7 \subset c$. The simplified version of the composition is as follows:

$$[Horizontal(b_2,a) \wedge Vertical(b_2,a)] \wedge$$
$$\big[[WeakNorth(c,b_2) \vee Horizontal(c,b_2) \vee WeakSouth(c,b_2)] \wedge [WeakEast(c,b_2) \vee Vertical(c,b_2) \vee WeakWest(c,b_2)]\big]$$
$$\equiv \big[[Horizontal(b_2,a)] \wedge [WeakNorth(c,b_2) \vee Horizontal(c,b_2) \vee WeakSouth(c,b_2)]\big] \wedge \big[[Vertical(b_2,a)] \wedge [WeakEast(c,b_2) \vee Vertical(c,b_2) \vee WeakWest(c,b_2)]\big]$$
$$= \big[[WeakNorth(c,a) \vee Horizontal(c,a) \vee WeakSouth(c,a)] \wedge [WeakEast(c,a) \vee Vertical(c,a) \vee WeakWest(c,a)]\big]$$

The final outcome of the composition is part(1)$\wedge$part(2) is equivalent to:

$$[WeakNorth(c,a) \vee Horizontal(c,a) \vee WeakSouth(c,a)] \wedge [WeakEast(c,a) \vee Vertical(c,a) \vee WeakWest(c,a)]$$

This means that the region $c \subseteq U$ which is the union of all the 9 tiles of region *a*. However, based on Figure 4, region $c \not\subset$ SW(*a*).

Composition of Expressive Direction Relations

We shall use the following notations to represent the 13 binary *y*-direction relations:

- *REL1y(b,a)*–NTPPy(b,WeakNorth(*a*))
- *REL2y(b,a)*–TPPy(b,WeakNorth(*a*))$\wedge$ECy(b,Horizontal(*a*))
- *REL3y(b,a))*–TPPy(b,Horizontal(*a*))$\wedge$ECy(b,WeakNorth(*a*))
- *REL4y(b,a))*–TPPy(b,Horizontal(*a*))$\wedge$ECy(b,WeakSouth(*a*))
- *REL5y(b,a)*–NTPPy(b,Horizontal(*a*))
- *REL6y(b,a)*–EQy(b,Horizontal(*a*))
- *REL7y(b,a)*–NTPPy(b,WeakSouth(*a*))
- *REL8y(b,a)*–TPPy(b,WeakSouth(*a*))$\wedge$ECy(b,Horizontal(*a*))
- *REL9y(b,a)*–POy(b,WeakNorth(*a*))$\wedge$POy(b,Horizontal(*a*))$\wedge$ DCy(b,WeakSouth(*a*))
- *REL10y(b,a)*–POy(b,WeakNorth(*a*))$\wedge$POy(b,Horizontal(*a*))$\wedge$ ECy(b,WeakSouth(*a*))
- *REL11y(b,a)*–POy(b,WeakNorth(*a*))$\wedge$POy(b,WeakSouth(*a*)) $\wedge$NTPPiy(b,Horizontal(*a*))
- *REL12y(b,a)*–POy(b,WeakSouth(*a*))$\wedge$POy(b,Horizontal(*a*))$\wedge$ DCy(b,WeakNorth(*a*))
- *REL13y(b,a)*–POy(b,WeakSouth(*a*))$\wedge$POy(b,Horizontal(*a*))$\wedge$ ECy(b,WeakNorth(*a*))

Similar notations will be used to represent the 13 binary *x*-direction relations (*WeakNorth* is replaced by *WeakEast*, *Horizontal* with *Vertical* and *WeakSouth* by *WeakWest*).

Example 1:

Find the composition of the following:

$$\left[{}^{REL3y(b,a)}\left[P_O(b,a)\right]_{REL3x(b1,a)} \wedge {}^{REL2y(b,a)}\left[PN_E(b,a)\right]_{REL2x(b1,a)}\right] \wedge \left[{}^{REL3y(c,b)}\left[A_N(c,b)\right]_{REL3x(c,)}\right]$$

Establish the direction relation between c and each individual part of b. Use Equation 2.b, with $1 \le k \le 2$ and $1 \le m_1 \le 2$, and $1 \le m_2 \le 2$.

$$\left[\left[P_{R1}(b_1,a)\right] \wedge \left[P_{R11}(c_1,b_1) \vee P_{R12}(c_2,b_1)\right]\right] \wedge \left[\left[P_{R2}(b_2,a)\right] \wedge \left[P_{R21}(c_1,b_2) \vee P_{R22}(c_2,b_2)\right]\right]$$

Use Equation (1), and the above composition can be rewritten in the following expressive form:

$$\left[\left[REL3y(b_1,a)\wedge REL3x(b_1,a)\right]\right]\wedge\left[\left[REL1y(c_1,b_1)\wedge REL3x(c_1,b_1)\right] \vee\left[REL1y(c_2,b_1)\wedge REL2x(c_2,b_1)\right]\right]\wedge$$
$$\left[\left[REL2y(b_2,a)\wedge REL2x(b_2,a)\right]\right]\wedge\left[\left[REL2y(c_1,b_2)\wedge REL4x(c_1,b_2)\right] \vee\left[REL2y(c_2,b_2)\wedge REL8x(c_2,b_2)\right]\right]$$
$$\equiv \left[REL3y(b_1,a)\wedge\left[REL1y(c_1,b_1)\vee REL1y(c_2,b_1)\right]\right]\wedge \left[REL3x(b_1,a)\wedge\left[REL3x(c_1,b_1)\vee REL2x(c_2,b_1)\right]\right]\wedge$$
$$\left[REL2y(b_2,a)\wedge\left[REL2y(c_1,b_2)\vee REL2y(c_2,b_2)\right]\right]\wedge \left[REL2x(b_2,a)\wedge\left[REL4x(c_1,b_2)\vee REL8x(c_2,b_2)\right]\right]$$

Use Tables 1 and 2, and $c_1 \subset c$ and $c_2 \subset c$. Thus, the outcome of the composition can be written as follows:

$$REL1y(c,a)\wedge\left[REL2x(c,a)\vee REL3x(c,a)\right]\wedge$$
$$REL1y(c,a)\wedge\left[REL2x(c,a)\vee REL3x(c,a) \vee REL6x(c,a)\vee REL13x(c,a)\right]$$
$$= REL1y(c,a)\wedge\left[REL2x(c,a)\vee REL3x(c,a)\right]$$

The outcome of the composition is:

NTPPy(c,WeakNorth(a))$\wedge$ [TPPx(c,WeakEast(a))$\wedge$ECy(c,Horizontal(a))$\vee$ TPPx(c,Vertical(a))$\wedge$ECy(c, Horizontal(a))]

Example 2:

This example is similar to the fourth example in the previous section of this paper.

Find the composition of

$[P_O(b_1,a) \wedge P_{NE}(b_2,a)] \wedge [P_O(c,b) \wedge P_W(c,b) \wedge P_{SW}(c,b)]$

Establish the direction relation between c and each individual part of b. Use Equation 2.b, with $1 \le k \le 2$ and $1 \le m_1 \le 4$, and $1 \le m_2 \le 7$.

The composition in expressive form will be as follows:

For part b_1

$$\left[\left[REL2y(b_1,a)\wedge REL2x(b_1,a)\right]\right]\wedge$$
$$\left[\left[REL4y(c_1,b_1)\wedge REL4x(c_1,b_1)\right]\vee\left[REL8y(c_2,b_1)\wedge REL4x(c_2,b_1)\right]\vee \left[REL4y(c_3,b_1)\wedge REL8x(c_3,b_1)\right]\vee\left[REL8y(c_4,b_1)\wedge REL8x(c_4,b_1)\right]\right]$$

The regions $c_1, c_2, c_3, c_4 \subset c$, the above composition can be written as follows:

$$\left[REL2y(b_1,a)\wedge\left[REL4y(c,b_1)\vee REL8y(c,b_1) \vee REL4y(c,b_1)\vee REL8y(c,b_1)\right]\right]\wedge$$
$$\left[REL2x(b_1,a)\wedge\left[REL4x(c,b_1)\vee REL4x(c,b_1) \vee REL8x(c,b_1)\vee REL8x(c,b_1)\right]\right]$$
$$= \left[REL2y(c,a)\vee REL3y(c,a)\vee REL6y(c,a)\vee REL13y(c,a)\right]\wedge \left[REL2x(c,a)\vee REL3x(c,a)\vee REL6x(c,a)\vee REL13x(c,a)\right]\ldots\text{(3a)}$$

For part b_2

$$\left[\left[REL3y(b_2,a)\wedge REL3x(b_2,a)\right]\right]\wedge$$
$$\left[\left[REL8y(c_1,b_2)\wedge REL7x(c_1,b_2)\right]\vee\left[REL6y(c_2,b_2)\wedge REL8x(c_2,b_2)\right]\vee \left[REL2y(c_3,b_2)\wedge REL8x(c_3,b_2)\right]\vee\left[REL2y(c_4,b_2)\wedge REL6x(c_4,b_2)\right]\vee \left[REL3y(c_5,b_2)\wedge REL6x(c_5,b_2)\right]\vee\left[REL3y(c_6,b_2)\wedge REL2x(c_6,b_2)\right]\vee \left[REL2y(c_7,b_2)\wedge REL2x(c_7,b_2)\right]\right]$$
$$= \left[REL2y(c,a)\vee REL3y(c,a)\vee REL5y(c,a)\vee REL12y(c,a)\right]\wedge \left[REL2x(c,a)\vee REL3x(c,a)\vee REL4x(c,a)\vee REL5x(c,a)\vee REL7x(c,a)\vee REL8x(c,a)\vee REL12x(c,a)\right]\ldots\text{(3b)}$$

The final outcome of the composition is the composition of part b_1 (Equation 3.a) $\wedge$ part b_2 (Equation 3.b).

Apply Rule 3 from the earlier part of the paper and we will get the following:

$$\left[REL2y(c,a)\vee REL3y(c,a)\vee REL6y(c,a)\vee REL13y(c,a)\right]\wedge \left[REL2y(c,a)\vee REL3y(c,a)\vee REL12y(c,a)\right]\wedge$$
$$\left[REL2x(c,a)\vee REL3x(c,a)\vee REL4x(c,a)\vee REL8x(c,a)\vee REL12x(c,a)\right]\wedge$$
$$\left[REL2x(c,a)\vee REL3x(c,a)\vee REL6x(c,a)\vee REL13x(c,a)\right]\ \ldots\text{(4)}$$

We collapse some of the disjunction of relations:

$REL6y(c,a)\vee REL13y(c,a) = REL13y(c,a)$

$REL4x(c,a)\vee REL8x(c,a)\vee REL12x(c,a) = REL12y(c,a)$

$REL6x(c,a)\vee REL13x(c,a) = REL13x(c,a)$

Equation 4 becomes:

$$\left[REL2y(c,a)\vee REL3y(c,a)\vee REL13y(c,a)\right]\wedge \left[REL2y(c,a)\vee REL3y(c,a)\vee REL12y(c,a)\right]\wedge$$
$$\left[REL2x(c,a)\vee REL3x(c,a)\vee REL12x(c,a)\right]\wedge \left[REL2x(c,a)\vee REL3x(c,a)\vee REL13x(c,a)\right]\ \ldots\text{(4.a)}$$

Region c is single-piece. Therefore, Equation 4.a becomes:

[POy(c,WeakNorth(a))$\wedge$POy(c,WeakSouth(a)) $\wedge$NTPPiy(c,Horizontal(a))]$\wedge$

[POx(c,WeakEast(a))$\wedge$POx(c,WeakWest(a)) $\wedge$NTPPix(c,Vertical(a))]

This means that the region $c \subseteq U$ which is the union of all the 9 tiles of region a. As mentioned earlier, based on Figure 4, region $c \not\subset$ SW(a). Thus, the outcome of the composition for weak relations (in the previous section) yields the same result as this composition. However, the computation for the latter is more tedious and complex when involving regions with many parts.

Existential Inference

In this paper, we shall demonstrate how the expressive hybrid cardinal direction model could be used to make several existential inferences which are not possible in existing models..

Example 1: Find *R(b,a)* such that *c* ⊂ WeakNorth(*b*) and *c* ⊂ WeakNorth(*a*)

Skiadopoulos model (2001) which is not expressive enough, cannot answer such query because the composition table computed is not existential. To answer this query, we must first specify the expressive relation between *a* and *c*.

There are two possible relations: TPPy(*c*,WeakNorth(*a*)) or *WeakNorth(c,a)*. If it is the former then composition is is *WeakNorth(b,a)*∧*WeakNorth(c,b)* which means *R(b,a)* is *WeakNorth(b,a)*. If it is the latter, there are several combinations:

- *WeakNorth(b,a)*∧*Horizontal(c,b)*
- *WeakNorth(b,a)*∧*WeakSouth(c,b)*
- *Horizontal(b,a)*∧*WeakNorth(c,b)*
- *WeakSouth(b,a)*∧*WeakNorth(c,b)*

The first relation in each composition listed above is *R(b,a)*.

Example 2: Find *R(b,a)* and *S(c,b)* such that *T(a,c)* is ¬[TPPy(*c*,Horizontal(*a*)) ∧ ECy(*c*,WeakSouth(*a*))**]**

Based on Table 1, 9 different compositions will yield the following outcome:

TPPy(*c*,Horizontal(*a*))∧ECy(*c*,WeakSouth(*a*))**]**

The set of possible compositions, *Q*, is:

{*REL1y(b,a)*∧*REL7y(c,b), REL2y(b,a)*∧*REL7y(c,b), REL3y(b,a)*∧*REL7y(c,b), REL3y(b,a)*∧*REL8y(c,b), REL5y(b,a)*∧*REL7y(c,b), REL5y(b,a)*∧*REL8y(c,b), REL6y(b,a)*∧*REL4y(c,b), REL7y(b,a)*∧*REL1y(c,b), REL8y(b,a)*∧*REL12(c,b)*}

If *U* equals 8 x 8 atomic binary direction relations, then the set of all possible ordered pairs of *R* and *S* which satisfy the above query will be *U – Q*.

Example 3: Find *R(b,a)* and *S(c,b)* such that *T(a,c)* is POy(*c*,WeakSouth(*a*))∧POy(*c*,Horizontal(*a*))∧ ECy(*c*,WeakNorth(*a*))

Based on Table 1, we have 4 pairs of *R* and *S* which satisfy *T*. They are: *REL1y(b,a)*∧*REL7y(c,b),REL2y(b,a)*∧ *REL8y(c,b), REL7y(b,a)*∧*REL1y(c,b), REL7y(b,a)*∧*REL2y(c,b).*

Conclusion

In this paper, we have shown how topological and direction relations can be integrated to produce a more expressive hybrid model for cardinal directions. The composition table derived from this model could be used to infer both weak and expressive direction relations between regions. We have also introduced and demonstrated how to use a formula to compute the composition of weak or expressive relations between 'whole and part' regions. We have also demonstrated how the composition table with expressive direction relations could be used to make several difficult existential inferences.

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<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">WeakNorth(c,b)</td><td colspan="4">Horizontal(c,b)</td><td colspan="2">WeakSouth(c,b)</td></tr>
<tr><td>NTPPy(c,WeakNorth(b))</td><td>TPPy(c,WeakNorth(b)) ∧ ECy(c,Horizontal(b))</td><td>TPPy(c,Horizontal(b)) ∧ECy(c,WeakNorth(b))</td><td>TPPy(c,Horizontal(b)) ∧ECy(c,WeakSouth(b))</td><td>NTPPy(c,Horizontal(b))</td><td>EQy(c,Horizontal(b))</td><td>NTPPy(c,WeakSouth(b))</td><td>TPPy(c,WeakSouth(b)) ∧ECy(c,Horizontal(b))</td></tr>
<tr><td rowspan="3">WeakNort(b,a)</td><td>NTPPy(b,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>U – 13 relations</td><td>NTPPy(c,WeakNorth(a))
∨
TPPy(c,WeakNorth(a))
∧ ECy(c,Horizontal(a))
∨
POy(c,WeakNorth(a)) ∧
POy(a,Horizontal(a)) ∧
DCy(c,WeakSouth(a))
∨
POy(c,WeakNorth(a)) ∧
POy(a,Horizontal(a)) ∧
ECy(c,WeakSouth(a))
∨
POy(c,WeakNorth(a)) ∧
POy(a,WeakSouth(a)) ∧
NTPPiy(c,Horizontal(a))</td></tr>
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TPPy(c,Horizontal(a)) ∧
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<tr><td rowspan="2">Horizontal(b,a)</td><td>TPPy(b,Horizontal(a))
∧
ECy(b,WeakNorth(a))</td><td>NTPPy(c,WeakNorth(a))</td><td>TPPy(c,WeakNorth(a)) ∧ECy(c,Horizontal(a))</td><td>TPPy(c,Horizontal(a)) ∧ ECy(c,WeakNorth(a))</td><td>NTPPy(c,Horizontal(a))</td><td>NTPPy(c,Horizontal(a))</td><td>TPPy(c,Horizontal(a)) ∧ ECy(c,WeakNorth(a))</td><td>NTPPy(c,Horizontal(a))
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</table>

		WeakNorth(c,b)		*Horizontal(c,b)*				*WeakSouth(c,b)*	
		NTPPy(c,WeakNorth(b))	**TPPy(c,WeakNorth(b)) ∧ ECy(c,Horizontal(b))**	**TPPy(c,Horizontal(b)) ∧ECy(c,WeakNorth(b))**	**TPPy(c,Horizontal(b)) ∧ECy(c,WeakSouth(b))**	**NTPPy(c,Horizontal(b))**	**EQy(c,Horizontal(b))**	**NTPPy(c,WeakSouth(b))**	**TPPy(c,WeakSouth(b)) ∧ECy(c,Horizontal(b))**
Horizontal(b,a)	**NTPPy(b,Horizontal(a))**	NTPPy(c,Horizontal(a)) ∨ TPPy(c,Horizontal(a)) ∧ ECy(c,WeakNorth(a)) ∨ TPPy(c,WeakNorth(a)) ∧ ECy(c,Horizontal(a)) ∨ NTPPy(c,WeakNorth(a)) ∨ POy(c,WeakNorth(a)) ∧ POy(a,Horizontal(a)) ∧ DCy(c,WeakSouth(a))	NTPPy(c,Horizontal(a)) ∨ TPPy(c,Horizontal(a)) ∧ ECy(c,WeakNorth(a)) ∨ POy(c,WeakNorth(a)) ∧ POy(a,Horizontal(a)) ∧ DCy(c,WeakSouth(a))	NTPPy(c,Horizontal(a))	NTPPy(c,Horizontal(a))	NTPPy(c,Horizontal(a))	NTPPy(c,Horizontal(a))	NTPPy(c,Horizontal(a)) ∨ TPPy(c,Horizontal(a)) ∧ ECy(c,WeakSouth(a)) ∨ TPPy(c,WeakSouth(a)) ∧ ECy(c,Horizontal(a)) ∨ NTPPy(c,WeakSouth(a)) ∨ POy(c,WeakSouth(a)) ∧ POy(a,Horizontal(a)) ∧ DCy(c,WeakNorth(a))	NTPPy(c,Horizontal(a)) ∨ TPPy(c,Horizontal(a)) ∧ ECy(c,WeakSouth(a)) ∨ POy(c,WeakSouth(a)) ∧ POy(a,Horizontal(a)) ∧ DCy(c,WeakNorth(a))
	EQy(b,Horizontal(a))	NTPPy(c,WeakNorth(a))	TPPy(c,WeakNorth(a)) ∧ECy(c,Horizontal(a))	TPPy(c,Horizontal(a)) ∧ ECy(c,WeakNorth(a))	TPPy(c,Horizontal(a)) ∧ ECy(c,WeakSouth(a))	NTPPy(c,Horizontal(a))	EQy(c,Horizontal(a))	NTPPy(c,WeakSouth(a))	TPPy(c,WeakSouth(a)) ∧ ECy(c,Horizontal(a))
	Horizontal(b,a)	*WeakNorth(c,a), ∨ Horizontal(c,a)*		*Horizontal(c,a)*				*Horizontal(c,a) ∨ WeakSouth(c,a)*	
WeakSouth(b,a)	**NTPPy(b,WeakSouth(a))**	U – 13 relations	NTPPy(c,WeakSouth(a)) ∨ TPPy(c,WeakSouth(a)) ∧ ECy(c,Horizontal(a)) ∨ POy(c,WeakSouth(a)) ∧ POy(a,Horizontal(a)) ∧ DCy(c,WeakNorth(a)) ∨ POy(c,WeakSouth(a)) ∧ POy(a,Horizontal(a)) ∧ ECy(c,WeakNorth(a)) ∨ POy(c,WeakSouth(a)) ∧ POy(a,WeakNorth(a)) ∧ NTPPiy(c,Horizontal(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))
	TPPy(b,WeakSouth(a)) & ECy(b,Horizontal(a))	NTPPy(c,Horizontal(a)) ∨ TPPy(c,Horizontal(a)) ∧ ECy(c,WeakNorth(a)) ∨ TPPy(c,WeakNorth(a)) ∧ ECy(c,Horizontal(a)) ∨ NTPPy(c,WeakNorth(a)) ∨ POy(c,WeakNorth(a)) ∧ POy(a,Horizontal(a)) ∧ DCy(c,WeakSouth(a))	TPPy(c,Horizontal(a)) ∧ ECy(c,WeakSouth(a)) ∨ EQy(c,Horizontal(a)) ∨ POy(c,WeakNorth(a)) ∧ POy(a,Horizontal(a)) ∧ ECy(c,WeakSouth(a))	TPPy(c,WeakSouth(a)) ∧ ECy(c,Horizontal(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))	TPPy(c,WeakSouth(a)) ∧ ECy(c,Horizontal(a))	NTPPy(c,WeakSouth(a))	NTPPy(c,WeakSouth(a))
		WeakNorth(c,a), ∨ Horizontal(c,a) ∨ WeakSouth(c,a),		*WeakSouth(c,a),*				NTPPy(c,WeakSouth(a))	

Note: The shaded boxes are for weak composition of relations

Table 1: Composition of binary relations in the *y*-direction for the hybrid model

		WeakEast(c,b)		***Vertical(c,b)***				***WeakWest(c,b)***	
		NTPPx(c,WeakEast(b))	**TPPx(c,WeakEast(b)) ∧ECx(c,Vertical(b))**	**TPPx(c,Vertical(b)) ∧ECx(c,WeakEast(b))**	**TPPx(c,Vertical(b)) ∧ECx(c,WeakWest(b))**	**NTPPx(c,Vertical(b))**	**EQx(c,Vertical(b))**	**NTPPx(c,WeakWest(b))**	**TPPx(c,WeakWest(b)) ∧ECx(c,Vertical(b))**
WeakEast(b,a)	**NTPPx(b,WeakEast(a))**	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	U – 13 relations	NTPPx(c,WeakEast(a)) ∨ TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakWest(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,WeakWest(a)) ∧ NTPPix(c,Vertical(a))
	TPPx(b,WeakEast(a)) ∧ ECx(b,Vertical(a))	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	NTPPx(c,WeakEast(a))	TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a))	NTPPx(c,WeakEast(a))	TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a)) ∨ NTPPx(c,WeakWest(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakEast(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ EQx(c,Vertical(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ ECx(c,WeakEast(a))
	WeakEast(b,a)	NTPPx(c,WeakEast(a))		***WeakEast(c,a)***				***WeakEast(c,a)* ∨ *Vertical(c,a)* ∨ *WeakWest(c,a)***	
Vertical(b,a)	**TPPx(b,Vertical(a)) ∧ ECx(b,WeakEast(a))**	NTPPx(c,WeakEast(a))	TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a))	NTPPx(c,Vertical(a))	NTPPx(c,Vertical(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a)) ∨ NTPPx(c,WeakWest(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakEast(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakEast(a))
	TPPx(b,Vertical(a)) ∧ ECx(b,WeakWest(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a)) ∨ NTPPx(c,WeakEast(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakWest(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakWest(a))	NTPPx(c,Vertical(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a))	NTPPx(c,Vertical(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a))	NTPPx(c,WeakWest(a))	TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a))

		WeakEast(c,b)		***Vertical(c,b)***				***WeakWest(c,b)***	
		NTPPx(c,WeakEast(b))	**TPPx(c,WeakEast(b))** **∧** **ECx(c,Vertical(b))**	**TPPx(c,Vertical(b))** **∧** **ECx(c,WeakEast(b))**	**TPPx(c,Vertical(b))** **∧** **ECx(c,WeakWest(b))**	**NTPPx(c,Vertical(b))**	**EQx(c,Vertical(b))**	**NTPPx(c,WeakWest(b))**	**TPPx(c,WeakWest(b))** **∧** **ECx(c,Vertical(b))**
Vertical(b,a)	**NTPPx(b,Vertical(a))**	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a)) ∨ NTPPx(c,WeakEast(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakWest(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakWest(a))	NTPPx(c,Vertical(a))	NTPPx(c,Vertical(a))	NTPPx(c,Vertical(a))	NTPPx(c,Vertical(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a)) ∨ NTPPx(c,WeakWest(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakEast(a))	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakEast(a))
	EQx(b,Vertical(a))	NTPPx(c,WeakEast(a))	TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a))	NTPPx(c,Vertical(a))	EQx(c,Vertical(a))	NTPPx(c,WeakWest(a))	TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a))
	Vertical(b,a)	***WeakEast(c,a) ∨ Vertical(c,a)***		***Vertical(c,a)***				***Vertical(c,a) ∨ WeakWest(c,a)***	
WeakWest(b,a)	**NTPPx(b,WeakWest(a))**	U – 13 relations	NTPPx(c,WeakWest(a)) ∨ TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakEast(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ POx(c,WeakWest(a)) ∧ POx(a,WeakEast(a)) ∧ NTPPix(c,Vertical(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))
	TPPx(b,WeakWest(a)) & **ECx(b,Vertical(a))**	NTPPx(c,Vertical(a)) ∨ TPPx(c,Vertical(a)) ∧ ECx(c,WeakEast(a)) ∨ TPPx(c,WeakEast(a)) ∧ ECx(c,Vertical(a)) ∨ NTPPx(c,WeakEast(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ DCy(c,WeakWest(a))	TPPx(c,Vertical(a)) ∧ ECx(c,WeakWest(a)) ∨ EQx(c,Vertical(a)) ∨ POx(c,WeakEast(a)) ∧ POx(a,Vertical(a)) ∧ ECx(c,WeakWest(a))	TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))	TPPx(c,WeakWest(a)) ∧ ECx(c,Vertical(a))	NTPPx(c,WeakWest(a))	NTPPx(c,WeakWest(a))
	WeakWest(b,a)	***WeakEast(c,a) ∨ Vertical(c,a) ∨ WeakWest(c,a)***		***WeakWest(c,a)***				NTPPx(c,WeakWest(a))	

Note: The shaded boxes are for weak composition of relations

Table 2: Composition of binary relations in the *x*-direction for the hybrid model